CLASSICAL ('ONTOLOGICAL') DUAL STATES IN QUANTUM THEORY AND THE MINIMAL GROUP REPRESENTATION HILBERT SPACE

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Abstract

We investigate the classical aspects of Quantum theory and under which description Quantum theory does *appear Classical*. Although such descriptions or variables are known as "ontological" or "hidden", they are *not hidden* at all, but are *dual classical* states (in the sense of the general classical-quantum duality of Nature). We analyze and interpret the dynamical scenario in an inherent quantum structure: **(i)** We show that the use of the known $|\varphi\rangle$ states in the circle (London 1926, 't Hooft 2024), takes a true dimension *only* when the system is subjected to the *minimal group representation* action of the metaplectic group $Mp(n)$. $Mp(n)$ Hermitian structure fully covers the symplectic $Sp(n)$ group and in certain cases $OSp(n)$. **(ii)** We compare the circle $|\varphi\rangle$ states and the the cilinder $|\xi\rangle$ states *in configuration space* with the two sectors of the full $Mp(2)$ Hilbert space corresponding to the *even* and *odd* n harmonic oscillators and their *total* sum. **(iii)** We compute the projections of the $Mp(2)$ states on the circle $\langle\varphi|$ and on the cilinder $\langle\xi|$ states. The known London circle states are *not normalizable.* We compute here the *general* coset coherent states $\langle\alpha,\varphi|$ in the circle, α being the coherent complex parameter: It allows fully *normalizability* of the complete set of the circle states. **(iv)** The London states (ontological in t'Hooft's description) *classicalize completely* the inherent quantum structure *only* under the action of the $Mp(n)$ *Minimal Group Representation* . **(v)** For the coherent states in the cylinder (configuration space), all functions are analytic in the disc $|\,z = \omega\, e^{-i\varphi}| < 1$. For the general coset coherent states $|\alpha,\varphi\rangle$ in the circle, the complex variable is $z' = z\, e^{-i\,\alpha^*/2}$: the analytic function is modified by the complex phase $(\varphi - \alpha^*/2)$. **(vi)** The analiticity $|\,z'| = |z|\, e^{-\,\mathrm{Im}\,\alpha/2} < 1$ occurs with $\mathrm{Im}\,\alpha \neq 0$ because of normalizability, and $\mathrm{Im}\,\alpha > 0$ because of the Identity condition. The circle topology induced by the $\langle\alpha,\varphi|$ coset coherent state, also modifies the ratio of the disc due the displacement by the coset. **(vii)** For the coset coherent cilinder states *in configuration space*, the classicalization is stronger due to screening exponential factors $e^{-2\,n^2}$, $e^{-(\,2n+1/2\,)}$ and $e^{-(\,2n+1/2\,)^2}$ for large n arising in the $Mp(2)$ projections on them. The generalized Wigner function shows a bell-shaped distribution and *stronger classicalization* than the square norm functions. The application of the Minimal Group Representation immediately classicalizes the system, $Mp(2)$ emerging as the group of the *classical-quantum duality* symmetry.

I. INTRODUCTION AND RESULTS

Nature is Quantum. Classical states are contained in a Quantum description. Quantum theory is in full developpement, whatever be in its own concepts and extensions, interpretations, and yet to be understood effects and manifestations, or in new applications, Artificial Intelligency, Quantum computing, technological and fundamental experimental searches. See for example Refs [1], [2], [3] and Refs therein. Is not the purpose of this Introduction to review it here.

In this paper, without entering in all the issues of the interpretations of Quantum theory, we consider the subject of the Classical aspects of Quantum theory and under which description Quantum theory does *appear Classical.*

Usually, the variables providing such a description are known as "ontological" or 'local hidden variables', - although *they are not hidden at all*, and we agree with 't Hooft Ref [4] who recently considered these variables as *not hidden.*

Usually, such classical or "ontological" variables are assumed to describe the results of the real measurements performed on a given quantum system, (ontological meaning here essential or existential, conceptual or substantial, for an entity or representation).

The general Classical- Quantum (wave-particle, de Broglie) duality is one deep foundational property of quantum theory and does remain a crucial one. More recently , this concept have been extended to include gravity at the Planck scale and beyond it (classical-quantum gravity duality), which is general, and does not depend on the number or type of space-time dimensions, nor manifolds (compactified or not), or on other considerations.

The Classical "ontological or existing" states are in fact *Classical Dual states* and we do prefer this last term because it is physically precise, appropriate and meaninful for these states.

The time evolution of the quantum harmonic oscillator does appear identical to the classical motion: a rigid rotating motion in phase space with the oscillator frequency. The quantum states of light are harmonic oscillators and have the same time evolution.

We also consider the *Wigner function* which is a useful tool for the comparison of the classical and quantum dynamics of these states in phase space, and to analyze the classical behavior in general. It allows to obtain the probability distribution for these states in phase space. The Wigner or quasiprobability distribution, namely $W(q,p)$ is symmetric in the reflection (time and space) symmetries, e.g $W(-q,-p) = W(q,p)$.

In this paper:

- In order to interpret the dynamical scenario connected with an inherent quantum structure, we show that the use of the known states in the circle ('t Hooft 2024 [4], London 1926 [5]), takes a true dimension only when the system is subjected to the minimal group representation under the action of the metaplectic group $Mp(n)$. Let us recall that $Mp(n)$ fully covers the symplectic group $Sp(n)$ and in certain cases its Hermitian structure can be extended to $OSp(n)$.

- We consider the Metaplectic group $Mp(2n)$, e.g the double covering of the symplectic group $Sp(2n)$. *Discretization* arises naturally and directly from the basic states of the metaplectic representations with an interesting feature to be highlighted here: the decomposition of the $SO(2,1)$ group into its two irreducible representations encompasses the span of the *even* $|\, 2n \,\rangle$ and *odd* $|\, 2n + 1 \,\rangle$ states, respectively, $(n = 1, 2, 3 \ldots)$ of the harmonic oscillator, whose entirety is covered by the metaplectic group. For $n \to \infty$, the spectrum becomes continuum as it must be.

- In the metaplectic representation, the general or complete states must be the sum of the two types states: *even and odd* n states, because they span respectively the two sectors of the Hilbert space, $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, completely covered by the Metaplectic symmetry group $M_p(2):\ \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$. Based on the highest eigenvalue of the number operator occurring in the complete Hilbert space, the two unitary irreducible representations of $Mp(2)$ are denoted $\mathcal{H}_{1/4}$ (even states) and $\mathcal{H}_{3/4}$ (odd states).

- We compare the complete and fully normalizable $Mp(2)$ states with the $|\varphi\rangle$ states in the circle (London states, phase space, and recently considered by 't Hooft).

- We also consider general coherent states $|\xi\rangle$ in the cilinder *in configuration space* and compare them with the sectors of the $Mp(2)$ Hilbert space, and with the *total* (sum of the two sectors) states (+) and (−), corresponding to the even ($2n$) and odd ($2n+1$) states of the harmonic oscillators. We compute the projections of the Mp(2) $|\,\Psi^{\pm}\,(\omega)\,\rangle$ states on the circle $\langle\,\varphi\,|$ and on the cilinder $\langle\,\xi\,|$ states.

- The known circle (London, t' Hooft) states are *coherent not normalizable states.* We compute here the *general* coherent states in the circle $\langle\,\alpha\,,\varphi\,|$ which include the complex parameter α characteristic of coherent states, whose meaning appears clearly allowing the fully *finite normalization* of the complete set of states. The power of the general *coset group construction procedure* of coherent states does show up here.

Main implications of the results of this paper are the following:

- **(i)** The London states (ontological in t'Hooft's description) *classicalize completely* the inherent quantum structure *only* under the application of the Minimal Group Representation with the $Mp(n)$ group taking the main role.

- **(ii)** The action of the Metaplectic group on the "ontological" (London) states breaks the invariance under time reversal assumed for the dynamics of the particle in the circle (arrow of time).

- **(iii)** In the case of the coherent states of a particle in the cylinder (configuration space), we can also assign to them the variable $z = \omega\, e^{-i\varphi}$ as in the case of the particle states in a circle, (London states, phase space). All functions are analytic in the disc $|\,z| < 1$. For the general coherent states in the circle $|\alpha, \varphi\rangle$ the complex variable is $z' = \omega\, e^{i\,(\varphi - \alpha^*/2)} = z\; e^{-i\,\alpha^*/2}$: the analytic function in the disc is modified by the complex phase $(\,\varphi - \alpha^*/2\,)$, α being the characteristic coherent state complex parameter.

- **(iv)** The analiticity condition on the disk : $|\,z'| = |z|\, e^{-\,\mathrm{Im}\,\alpha/2} < 1$ occurs with the condition $\mathrm{Im}\,\alpha \neq 0$ arising from the normalizability, and with $\mathrm{Im}\,\alpha > 0$ arising from the Identity condition. The topology of the circle induced by the coset coherent state $\langle\alpha, \varphi|$ not only modifies the phase of ω (e.g: $\omega\, e^{i\,(\varphi - \alpha^*/2)} = z'$) but also the ratio of the disc due the displacement generated by the action of the coset.

- **(v)** The norm of the projection of the $Mp(2)$ states on the cilinder and on general circle coherent states clearly shows a very fast decreasing as n increases or *classicalization.* In the case of the cilinder states *in configuration space* the decreasing is stronger than in the circle phase space states, due to the exponentials e^{-2n^2}, $e^{-(2n+1/2)}$ and $e^{-(2n+1/2)^2}$, arising in these configuration space projections.

The generalized Wigner function for the circle states displays a classicalized distribution, more bell- shaped that the square norm function of these states.

More remarks are presented in the Conclusions.

This paper is organized as follows: In Section II we describe the quantum dynamics on the circle and its phase space states (London states). In Section III we summarize the Metaplectic Minimal Group Representation approach, its group content, double covering and fully complete Hilbert space of states. In Section IV both the $Mp(2)$ basic states and the ontological states in the circle are compared together with their their mutual scalar products, which shows how classicalization does occurs in this case. In Section V we consider the general coherent states in a cilinder in configuration space and compute the projections of the $Mp(2)$ total states on them, which show a still more strong classicalization with respect to the London (circle, phase space) states. Section VI discusses the implications of the $Mp(2)$ Representation Reduction on theses states and the analysis and results of the previous sections. Sections VII and VIII the general normalized coset coherent states on the circle and the projected $Mp(2)$ Reduction on them are computed and analysed. Section IX summarizes the Conclusions.

II. "ONTOLOGICAL" STATES AND THE MINIMAL GROUP REPRESENTATION

As we showed earlier Ref [6] , there is an even more general principle in the fundamental structure of quantum spacetime: *the principle of minimal group representation*, which allows us to obtain, consistently and simultaneously, a natural description of the dynamics of spacetime and the physical states admissible within it.

The theoretical design is based on physical states which are the mean values of the metaplectic group generators $Mp(n)$, the double covering $SL(2C)$ in vector representation, relative to the coherent states bearing the spin weight.

In this theoretical context, there is a connection between the dynamics given by the symmetry generators of the metaplectic group and the physical states (mappings of the generators through bilinear combinations of the ground states), Refs [6], [7].

Let us now see how to apply this principle to the problem of the construction of classical variables in quantum theory as considered by t' Hooft in Ref [4]. Therefore, we first need to consider the quantum dynamics of a particle on the circle, which we describe in the next Section.

A. II Theoretical aspects of the quantum dynamics on the circle

(i) The starting point to have into account to describe a free particle on a circle is its Hamiltonian

$$H \; = \; L = \frac{1}{2}\dot{\varphi}^{2}(t) \quad \left(e.g: \;\; \dot{\varphi} \; = \; J \right)$$

here with unit mass and velocity, φ being the angle position, with period 2π:

$$\varphi\,(t) = \varphi(0) + t$$

Classically,

$$\{\varphi, J\} \; = \; 1$$

Quantically, (operator level):

$$\left[\widehat{\varphi}, \widehat{J}\right] \; = \; i \tag{1}$$

The best candidate for the position operator (well defined in Hilbert space) is:

$$U \; = \; e^{i\,\widehat{\varphi}} \qquad (\text{U is unitary})$$

And it is easy to see from the above equations that:

$$\left[\widehat{J}\;,\; U\right] \; = \; U \tag{2}$$

(ii) Let now consider the eigenstates

$$\widehat{J}\;|\;j\;\rangle \; = \; j\;|\;j\;\rangle \tag{3}$$

From Eqs (2) and (3) we have:

$$U\;|\;j\;\rangle \; = \; j\,|\;j+1\;\rangle\,,$$
$$U^{\dagger}\;|\;j\;\rangle = \; j\,|\;j-1\;\rangle$$

e.g. $U^{\dagger}$ and U are ladder operators.

Now, it is esily seen the additional properties the states $|\, j\,\rangle$ satisfy:

$$\langle\, i\,||\, j\,\rangle \;=\; \delta_{i\,j} \qquad \text{(orthogonality)}$$

$$\sum_j |\, j\,\rangle\langle\, j\,| \;=\; \mathbb{I} \qquad \text{(completeness)}$$

As we see, we have all the ingredients to implement the principle of minimal representation of the group:

- (i) the basis $|\, j\,\rangle$ *isomorphic to* the basis $|\, n\,\rangle$ of the harmonic oscillator,

- (ii) a *symplectic structure*, and (iii) the conmutator Eq.(1) that allows to build the two operators:

$$a \;=\; \frac{1}{\sqrt{2}}\left(\widehat{\varphi} \,+\, i\widehat{J}\right) \quad \text{and} \quad a^{+} = \frac{1}{\sqrt{2}}\left(\widehat{\varphi} \,-\, i\widehat{J}\right)$$

that is, a and a^{+} are ladder operators satisfying:

$$\left[a\;,\; a^{+}\right] \;=\; 1 \tag{4}$$

Let us now consider the Metaplectic group approach.

III. THE METAPLECTIC MINIMAL GROUP REPRESENTATION APPROACH

In Ref [6], [7] and refs. therein, a group-theoretic approach was developed to obtain the metric (line element) as the central geometrical object associated to a discrete quantum structure of the spacetime for a quantum theory of gravity.

In summary for the purpose of this paper, the main characteristics of this framework are the following:

- Such an emergent metric is obtained from a Riemmanian superspace and is described as a physical coherent state of the underlying cover of the $SL(2C)$ group: Interestingly, it seems necessary to consider the *full cover of the symplectic group*, which is the metaplectic group $Mp(n)$, its spectrum for all n leads in particular for very large n to continuous spacetime.

- The main and fundamental importance of this quantum description is based on the phase space of a relativistic particle in the *natural* superspace of bosonic and fermionic coordinates that allow preserving at the quantum level the square root forms of geometric operators (e.g. the Hamiltonian or the Lagrangian).

- Such natural characteristic of this description is ensured by a complete bosonic realization of the underlying algebra through creation and annihilation operators of the harmonic oscillator that establish the gradation of it.

- The *discrete* structure of the spacetime arises directly from the basic states of the metaplectic representation with an interesting feature to be highlighted here: the decomposition of the $SO(2,1)$ group into its two irreducible representations encompasses the span of *even* $\mid 2n \,\rangle$ and *odd* $\mid 2n \,+\, 1 \,\rangle$ states, ($n = 1,\, 2,\, 3\ldots$), respectively, whose entirety is covered by the metaplectic group.

- In the metaplectic representation, the general or complete states must be the sum of the two types of states: the even and odd n states, because they span respectively the two sectors of the Hilbert space, $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, whose complete covering is $\mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$ corresponding to the Metaplectic symmetry group $M_p(2)$.

- This yields the relativistic quantum metric of the discrete structure spacetime as the fundamental basis for a quantum theory of gravity. For increasing numbers of levels n, the metric solution goes to the continuum and to the classical general relativistic manifold as it should be.

- The double covering of *even and odd* n states and *their sum* in order to have the complete Hilbert space reflects here the CPT completeness of the theory, and such property is the reflection of unitarity. As we know, the metaplectic group $M_p(2)$ acts irreducibly on each of the subspaces $\mathcal{H}_{1/4}$, $\mathcal{H}_{3/4}$ (even and odd sectors) by which the total Hilbert space (i.e.,$\mathcal{H}$) is divided, according to the $M_p(2)$ Casimir operator which gives precisely the values $k \;=\; 1/4,\;\; 3/4$:

$$K^2 \;=\; K_3^2 \;-\; K_1^2 \;-\; K_2^2 \;=\; k\;(k\;-\;1)\;=\;-\,\frac{3}{16}\,\mathbb{I}$$

Therefore,

$$\mathcal{H}_{1/4} \;=\; \mathrm{Span}\,\{\,\mid n\ \textit{even}\,\rangle\ \text{states}:\ n \;=\; 0,\,2,\,4,\,6,\,...\,\} \tag{5}$$

$$\mathcal{H}_{3/4} \;=\; \mathrm{Span}\,\{\,\mid n\ \textit{odd}\,\rangle\ \text{states}:\ n \;=\; 1,\,3,\,5,\,7,\,....\,\} \tag{6}$$

Based on the highest eigenvalue of the number operator occurring in the complete $\mathcal{H} \equiv \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$:

$$T_3 \mid n \rangle = -\frac{1}{2}\left(n + \frac{1}{2}\right) \mid n \rangle ,$$

the two unitary irreducible representations (UIR) of $Mp(2)$ are denoted as:

$$(UIR) \quad \text{restricted to } \mathcal{H}_{1/4} \rightarrow \mathcal{D}_{1/4} \in Mp(2) \tag{7}$$

$$(UIR) \quad \text{restricted to } \mathcal{H}_{3/4} \rightarrow \mathcal{D}_{3/4} \in Mp(2) \tag{8}$$

It is notable that in the general case, $Sp(2m)$ can be embedded somehow in a larger algebra as $(Sp(2m) + R^{2m})$ admitting an Hermitian structure with respect to which it becomes the orthosymplectic superalgebra $Osp(2m, 1)$.

Consequently, the metaplectic representation of $Sp(2m)$ extends to an irreducible representation (IR) of $Osp(2m, 1)$ which can be realized in terms of the space H_h of all holomorphic functions h:

$$C^m \rightarrow C / \int |h(z)|^2 e^{-|z|^2} d\lambda(z) < \infty$$

with $\lambda(z)$ the Lebesgue measure on C^m.

The restriction of the $Mp(n)$ representation to $Sp(2m)$, implies that the two irreducible (even and odd n) sectors are supported by the subspaces $H_h^{\pm}$ of the holomorphic functions space H_h. H_h^+ and H_h^- are the (closed) spans of the set of functions:

$$z^n \equiv (z_1^{n_1}, ..., z_m^{n_m})$$

where $n_\theta \in Z$, $|n| = \sum n_\theta$, *even* and *odd*, (H_h^+ and H_h^-), respectively.

A. III Mp(2), SU(1,1) and Sp(2)

All the groups $Mp(2)$, $Sp(2, R)$, and $SU(1, 1)$ are three dimensional. It is possible to parameterize them in several ways that make the homomorphic relations particularly simple. We use two of such parameterizations, in terms of the $Mp(2)$ group generators (T_1, T_2, T_3) and the angles $(\alpha_1, \alpha_2, \alpha_3)$, both of which are described as:

$$Mp(2) \rightarrow e^{-i\alpha_1 T_1}, \quad e^{-i\alpha_2 T_2}, \quad e^{-i\alpha_3 T_3}$$

$$Sp\,(2R) \;\rightarrow\; \begin{pmatrix} e^{\frac{1}{2}\alpha_1} & 0 \\ 0 & e^{-\frac{1}{2}\alpha_1} \end{pmatrix}, \begin{pmatrix} \cosh\frac{1}{2}\alpha_2 & \sinh\frac{1}{2}\alpha_2 \\ \sinh\frac{1}{2}\alpha_2 & \cosh\frac{1}{2}\alpha_2 \end{pmatrix}, \begin{pmatrix} \cos\frac{1}{2}\alpha_3 & -\sin\frac{1}{2}\alpha_3 \\ \sin\frac{1}{2}\alpha_3 & \cos\frac{1}{2}\alpha_3 \end{pmatrix}$$

$$SU(1,1) \rightarrow \begin{pmatrix} \cosh\frac{1}{2}\alpha_1 & \sinh\frac{1}{2}\alpha_1 \\ \sinh\frac{1}{2}\alpha_1 & \cosh\frac{1}{2}\alpha_1 \end{pmatrix}, \begin{pmatrix} \cosh\frac{1}{2}\alpha_2 & i\sinh\frac{1}{2}\alpha_2 \\ -i\sinh\frac{1}{2}\alpha_2 & \cosh\frac{1}{2}\alpha_2 \end{pmatrix}, \begin{pmatrix} e^{\frac{i}{2}\alpha_3} & 0 \\ 0 & e^{-\frac{i}{2}\alpha_3} \end{pmatrix}$$

where the angle α_3 has the range $(-4\pi, 4\pi]$ for $Mp(2)$, and the range $(-2\pi, 2\pi]$ for $Sp\,(2, R)$ and $SU(1,1)$.

Let us consider the brief description of the relevant symmetry group to perform the realization of the Hamiltonian operator of the problem. This group specifically is the Metaplectic $Mp\,(2)$ as well as the groups that are topologically covered by it. The generators of $Mp\,(2)$ are the following :

$$\begin{aligned} T_1 &= \frac{1}{4}\,(\,q\,p \,+\, p\,q\,) \;=\; \frac{i}{4}\,\left(a^{+2} -\, a^2\right), \\ T_2 &= \frac{1}{4}\,\left(p^2 \,-\, q^2\right) \;=\; -\,\frac{1}{4}\,\left(a^{+2} +\, a^2\right), \\ T_3 &= -\,\frac{1}{4}\,\left(p^2 \,+\, q^2\right) \;=\; -\,\frac{1}{4}\,\left(a^{+}a \,+\, a\,a^{+}\right) \end{aligned} \tag{9}$$

with the commutation relations,

$$[\,T_1,\, T_2\,] \;=\; -\,i\,T_3$$

$$[\,T_3,\, T_1\,] \;=\; i\,T_2, \qquad [\,T_3,\, T_2\,] \;=\; -\,i\,T_1,$$

being $(q,\, p)$, or alternatively $(a,\, a^{+})$, the variables of the standard harmonic oscillator, as usual.

If we rewrite the commutation relations as:

$$[\,T_3\,,\, T_1 \,\pm\, i\,T_2\,] \;=\; \pm\,(\,T_1 \,\pm\, i\,T_2\,),$$

$$[\,T_1 \,+\, i\,T_2\,,\, T_1 \,-\, i\,T_2\,] \;=\; -\,2\,T_3,$$

we see that the states $|\,n\,\rangle$ are eigenstates of T_3:

$$T_3\;|\,n\,\rangle \;=\; -\,\frac{1}{2}\,\left(n+\frac{1}{2}\right)|\,n\,\rangle$$

And it is easy to see that:

$$T_1 \;+\; i\,T_2 \;=\; -\,\frac{i}{2}\,a^2, \qquad T_1 -\; i\,T_2 \;=\; \frac{i}{2}\,a^{+2}.$$

IV. THE MP(2) BASIC STATES VS. ONTOLOGICAL STATES IN THE CIRCLE

Let us look at the sectors $s = 1/4$ and $s = 3/4$ of the Hilbert space spanned by the $Mp(2)$ coherent states $\left| \Psi^{(\pm)}(\omega) \right\rangle$, ω being the frequency.

For the $s = 1/4$ sector, the basic state is:

$$\left| \Psi^{(+)}(\omega) \right\rangle = \left(1 - |\omega|^2\right)^{1/4} \sum_{n=0,1,2..} \frac{(\omega/2)^{2n}}{\sqrt{2n\,!}} \, |2n\rangle \, .$$

On the other hand, the ontological states in the circle (in the limit $N \to 1$) considered by t' Hooft, and previously proposed by London as phase states in Ref. [5] in terms of the eigenstates n of the harmonic oscillator are:

$$\langle\varphi| = \frac{1}{\sqrt{2\pi}} \sum_{n\,=\,0,1,2..} e^{i\varphi n} \langle n| \tag{10}$$

They are overcomplete:

$$\begin{aligned} \langle\varphi| |\varphi'\rangle &= \frac{1}{2\pi} \sum_{m\,=\,0,1,2...} \sum_{n\,=\,0,1,2...} e^{i\varphi n} e^{-i\varphi' m} \langle n| |m\rangle \\ &= \frac{1}{2\pi} \sum_{n\,=\,0,1,2...} e^{i(\varphi-\varphi')n} \end{aligned}$$

and solve the identity

$$\begin{aligned} \int_0^{2\pi} |\varphi\rangle \langle\varphi| \, d\varphi &= \underbrace{\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\varphi} \, d\varphi}_{\delta_{n,m}} \sum_{m\,=\,0,1,2...} \sum_{n\,=\,0,1,2...} |m\rangle \langle n| \\ &= \sum_{n\,=\,0,1,2...}^{\infty} |n\rangle \langle n| = \mathbb{I} \end{aligned}$$

Therefore, the scalar product of the two sets of states are:

$$\langle \varphi| \left|\Psi^{(+)}(\omega) \right\rangle = \frac{\left(1 - |\omega|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{m\,=\,0,1,2..,} \sum_{n\,=\,0,1,2..} \frac{(\omega/2)^{2n}}{\sqrt{2n\,!}} \, e^{i\varphi m} \langle m| |2n\rangle \tag{11}$$

$$\langle \varphi| \left|\Psi^{(+)}(\omega) \right\rangle = \frac{\left(1 - |\omega|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n\,=\,0,1,2..} \frac{(\omega \, e^{i\varphi}/2)^{2n}}{\sqrt{2n\,!}} \tag{12}$$

$$= \frac{\left(1 - |z|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n\,=\,0,1,2..} \frac{(z/2)^{2n}}{\sqrt{2n\,!}}, \qquad z = \omega \, e^{i\varphi}, \tag{13}$$

with $z = \omega e^{i\varphi}$, the analytic function in the disc is modified by the phase without changing the consistency of the conformal map.

Similarly, for the sector $s = 3/4$:

$$\langle \varphi| \left|\Psi^{(-)}(\omega)\right\rangle = \frac{\left(1-|\omega|^2\right)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{\left(\omega\, e^{i\varphi}/2\right)^{2n+1}}{\sqrt{(2n+1)\,!}} \tag{14}$$

$$= \frac{\left(1-|z|^2\right)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z/2)^{2n+1}}{\sqrt{(2n+1)\,!}} \tag{15}$$

Notice that by taking the scalar product between the ontological states $|\varphi\rangle$ and the states of $Mp(2)$, we obtain *two non-equivalent* expansions in terms of analytical functions on the disk for the sectors of the minimal representations $s = 1/4,\, 3/4$, corresponding to the *even* and *odd* n eigenstates of the harmonic oscillator respectively.

Consequently, $(\omega\, e^{i\varphi} = z)$:

$$\langle \varphi| \left|\Psi^{(\pm)}(z)\right\rangle = \begin{cases} \frac{\left(1-|z|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z/2)^{2n}}{\sqrt{(2n)\,!}} & (+):\ \text{even states} \\[2ex] \frac{\left(1-|z|^2\right)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z/2)^{2n+1}}{\sqrt{(2n+1)\,!}} & (\text{-}):\ \text{odd states} \end{cases} \tag{16a}$$

Therefore, the total or complete projected state $\langle \varphi| |\Psi(z)\rangle$ is given by:

$$\langle \varphi| |\Psi(z)\rangle = \langle \varphi| \left|\Psi^{(+)}(z)\right\rangle + \langle \varphi| \left|\Psi^{(-)}(z)\right\rangle \tag{17}$$

$$\langle \varphi || \Psi(z)\rangle = \frac{\left(1-|z|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z/2)^{2n}}{\sqrt{(2n)\,!}} \left[1 + \left(1-|z|^2\right)^{1/2} \frac{(z/2)}{\sqrt{2n+1}}\right] \tag{18}$$

Now let us consider the following observations about the total expression $\langle \varphi || \Psi(z)\rangle$:

- **(i)** The function $\langle\varphi| |\Psi(z)\rangle$ is analytic on the unit disk: $|z| = |z| < 1$.

- **(ii)** The topology of the circle induced by the state $\langle\,\varphi\,|$ Eq. (10) only modifies the phase of ω (e.g: $\omega\, e^{i\varphi} = z$) in the projection $\langle\,\varphi\,||\,\Psi\,(\omega)\,\rangle$ Eq. (18). This is so because the topology of the circle with $R = 1$ coincides with that of the unit disk.

- **(iii)** The norm of Eq. (18) is obtained giving as a result the function:

$$
\begin{aligned}
|\,\langle\,\varphi\,||\Psi\,(z)\,\rangle\,|^2 \;&=\; \frac{\left(1\;-\;|z|^2\right)^{1/2}}{2\pi}\sum_{n\,=\,0,1,2..}\frac{|z/2|^{4n}}{(2n)!}\left[\,1\;+\;\left(1-|z|^2\right)\frac{|z/2\,|^2}{2n+1}\,\right]\\
&=\;\frac{\left(1\;-\;|z|^2\right)^{1/2}}{2\pi}\sum_{n\,=\,0,1,2..}\left[\,\frac{|\,z/2\,|^{4n}}{(2n)!}\;+\;\left(1\;-\;|\,z\,|^2\right)^{3/2}\,\frac{|\,z/2\,|^{2\,(2n+1)}}{(2n+1)!}\,\right]\\
&=\;\left(1\;-\;|\,z\,|^2\,\right)^{1/2}\cosh\left(\,|\,z\,|^2/2\,\right)\;+\;\left(1\;-\;|\,z\,|^2\right)^{3/2}\sinh\left(\,|\,z\,|^2/2\,\right)
\end{aligned}
$$

that it is evidently analytic in the disc $|z| = |\omega| < 1$, graphically represented in the Figures 1 and 2.

V. GENERAL COHERENT STATES IN CONFIGURATION SPACE

In this section we will elucidate and clarify the concept of the existence of an underlying quantum structure in physical systems, in light of the results of the previous section. For this purpose, we implement the Minimal Group Representation approach to the case of the dynamics of a particle in the geometry of a cylinder. The dynamics of a particle on a cilinder have been studied in Refs [19], [20], [21], and in Ref [22] for the non-orientable case.

In order to introduce the Coherent States for a quantum particle on the cylinder geometry, it is possible to follow the Barut–Girardello construction and seek the Coherent State as the solution of the eigenvalue equation:

$$X\,|\xi\rangle \;=\; \xi|\,|\xi\rangle$$

with the complex parameter ξ, similarly to the standard case, and

$$X \;=\; e^{i\left(\widehat{\varphi}+\widehat{J}\right)}$$

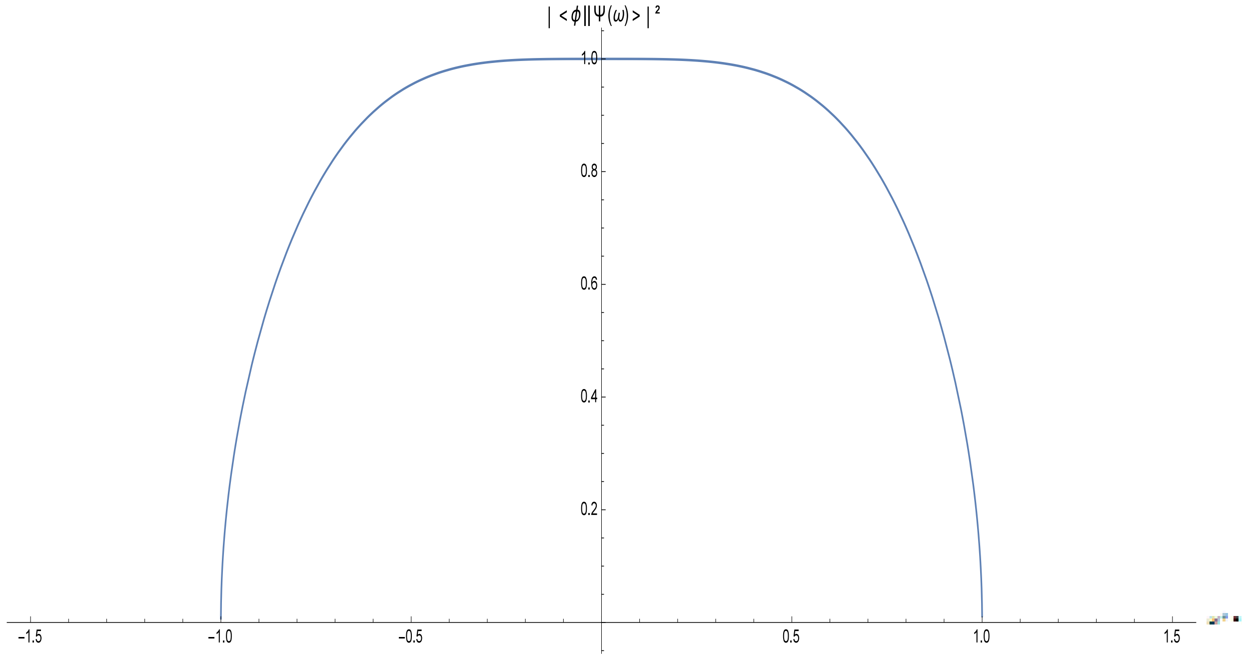

FIG. 1: Graphical representation of the the norm of the projection of the total Megaplectic $Mp(2)$ states onto the circle $|\varphi\rangle$ states: The function $|\langle\varphi|\,|\Psi\,(\omega)\rangle|^2$. As shown, analyticity is evident since it clearly respects $\left|z=\omega e^{i\varphi}\right|=|\omega|<1$.

In order to analyze the Coherent State of a particle in the cylinder in the context of the Minimal Group Representation, we express the coherent states as:

$$|\,\xi\,\rangle \;=\; \sum_{j\,=\,-\infty}^{\infty} e^{(l-i\varphi)j} e^{-j^2/2}\;|\,j\,\rangle \tag{19}$$

If $|\,j\,\rangle \sim |\,n\,\rangle$ and for the $s \;=\; 1/4$ Metaplectic states $\left|\,\Psi^{(+)}\,(\omega)\,\right\rangle$:

$$\langle\,\xi\,|\left|\,\Psi^{(+)}\,(\omega)\,\right\rangle = \left(1-|\omega|^2\right)^{1/4} \sum_{m\,=\,-\infty}^{\infty}\;\sum_{n\,=\,0,1,2...} \frac{(\omega/2)^{2n}}{\sqrt{2n\,!}}\; e^{(l-i\varphi)\,m}\, e^{-2m^2}\,\langle m|\,|2n\rangle$$

$$= \left(1-|\omega|^2\right)^{1/4} \sum_{n\,=\,0,1,2..} \frac{\left(\omega\, e^{\,(l-i\varphi)}/2\right)^{2n}}{\sqrt{2n\,!}}\; e^{-2n^2}$$

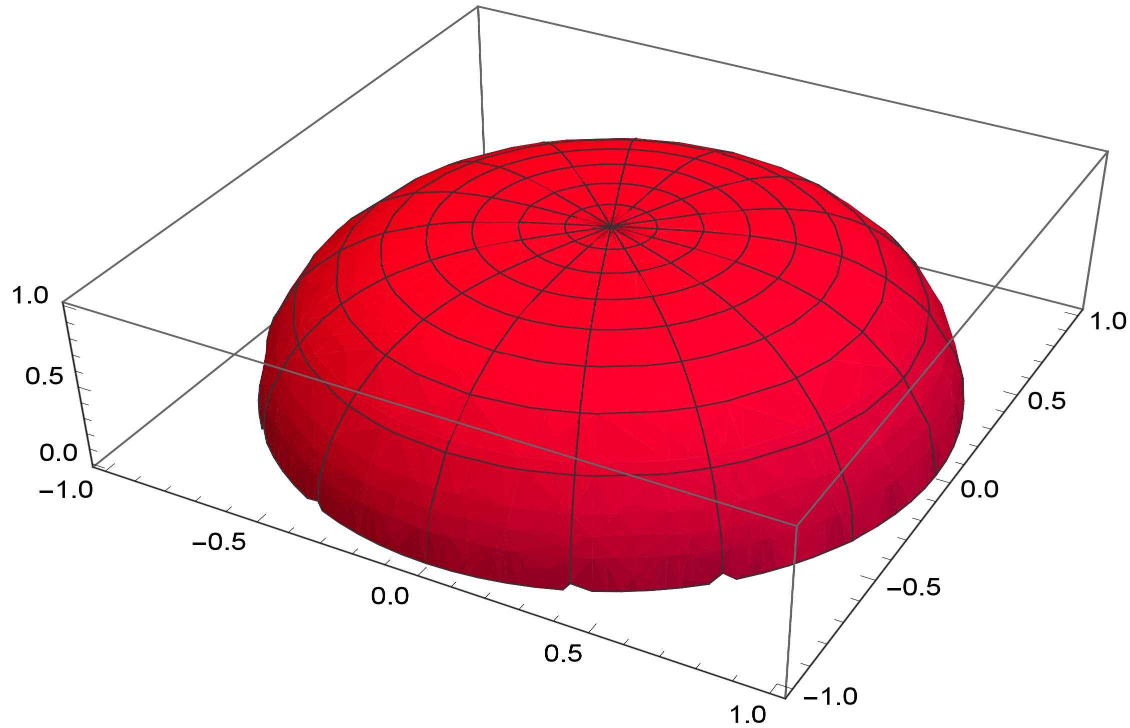

FIG. 2: Three-dimensional representation of the norm of the projection of the total Megaplectic $Mp(2)$ states on the circle $|\varphi\rangle$ states: The function $|\langle\varphi|\,|\Psi(\omega)\rangle|^2$, showing clearly the analytical character due to $\left|z=\omega\, e^{i\varphi}\right| = |\omega| < 1$.

Similarly, for the $s = 3/4$ Metaplectic states $\left|\,\Psi^{(-)}(\omega)\right\rangle$:

$$\langle\,\xi\,|\,\big|\,\Psi^{(-)}(\omega)\,\big\rangle = \left(1-|\omega|^2\right)^{3/4} \sum_{m=-\infty}^{\infty}\ \sum_{n\,=\,0,1,2\ldots} \frac{(\omega/2)^{2n+1}}{\sqrt{(2n+1)\,!}}\, e^{(l-i\varphi)\,m}\, e^{-m^2/2}\ \langle m|\,|2n+1\rangle$$

$$= \left(1-|\omega|^2\right)^{3/4} \sum_{n\,=\,0,1,2\ldots} \frac{\left(\omega\, e^{(l-i\varphi)}/2\right)^{2n+1}}{\sqrt{(2n+1)\,!}} e^{-(2n+1)^2/2}$$

We see that the scalar product projections taken with the cilinder $\langle\,\xi\,|$ space configuration states are similar to the projections taken with the circle $\langle\,\varphi\,|$ phase space states, but *in contrast* they contain weight functions: e^{-2n^2} and $e^{-(2n+1)^2/2}$, which *drastically attenuate* the scalar products when $n \to \infty$:

$$\langle\, \xi \,|\, \big|\, \Psi^{(\pm)}\,(\omega)\,\big\rangle = \begin{cases} \left(1-|\omega|^2\right)^{1/4} \displaystyle\sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{(l-i\varphi)}/2\right)^{2n}}{\sqrt{2n\,!}}\, e^{-2n^2} & \text{even states} \\ \left(1-|\omega|^2\right)^{3/4} \displaystyle\sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{(l-i\varphi)}/2\right)^{2n+1}}{\sqrt{(2n+1)\,!}}\, e^{-(2n+1)^2/2} & \text{odd states} \end{cases} \tag{20a}$$

Consequently, for the total state :

$$\big|\, \Psi\,(\omega)\,\big\rangle \;=\; \big|\, \Psi^{(+)}\,(\omega)\,\big\rangle \;+\; \big|\, \Psi^{(-)}\,(\omega)\,\big\rangle\,,$$

we have:

$$\langle\, \xi \,|\,|\, \Psi\,(\omega)\,\rangle = \left(1-|\omega|^2\right)^{1/4} \sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{(l-i\varphi)}/2\right)^{2n}}{\sqrt{2n\,!}}\, e^{-2\,n^2} \left[1+\left(1-|\omega|^2\right)^{1/2} \frac{\omega\, e^{\,(l-i\varphi)}}{\sqrt{2n+1}}\, e^{-(2n+1)/2}\right] \tag{21}$$

We discuss the implications of these results in the next Section.

VI. IMPLICATIONS OF THE MINIMAL GROUP REPRESENTATION REDUCTION

As we have seen so far, in order to interpret the dynamical scenario connected with an inherent quantum structure, the use of the London circle states takes a true dimension only when the system is subjected to the minimal group representation under the action of the metaplectic group $Mp(n)$. Let us recall that $Mp(n)$ covers $Sp(n)$ twice and in certain cases its Hermitian structure can be extended to $OSp(n)$.

Below we outline some implications that do result from the developments and analysis in the previous Sections.

- **(i)** The London states (ontological states in t'Hooft's description) *classicalize completely* the inherent quantum structure *only* under the application of the Minimal Group Representation with the $Mp(n)$ group taking the main role.

- **(ii)** The action of the Metaplectic group on the "ontological" (London) states breaks the invariance under time reversal assumed for the dynamics of the particle in the circle (arrow of time).

- **(iii)** In the case of the coherent states of a particle in the cylinder (configuration space) of Section V, we can set in our analysis the parameter $l = 0$: Thus we can also assign to them the variable $z = \omega\, e^{-i\varphi}$ as in the case of the particle states in a circle, (London states, phase space).

- **(iv)** In the case of the cilinder states, item (iii) with $l = 0$, the norm of the projection Eq. (21) is easily calculated giving as a result the function:

$$|\langle\, \xi\, |\, |\Psi\,(\omega)\,\rangle|^2 \;=\; \sum_{n\,=\,0,1,2...} \frac{|\,\omega/2\,|^{4n}}{2n\,!}\; e^{-2n^2}\, G\,(\omega,\varphi)$$

$$G\,(\omega,\varphi) \;=\; \left[1 \;+\; \frac{\left(1-|\omega|^2\right)^{1/2}}{\sqrt{2n+1}}\, e^{-2n-1/2} \left(2\,\mathrm{Re}\left(\omega\, e^{-i\varphi}\right) + e^{-2n-1/2}\left(1-|\omega|^2\right)^{1/2} \frac{|\omega/2|^2}{\sqrt{2n+1}}\right)\right]$$

where we see the very fast decreasing of the function due to the exponentials e^{-2n^2}, $e^{-(2n+1/2)}$ and $e^{-(2n+1/2)^2}$, arising in the projections of the Metaplectic states $|\Psi\,(\omega)\,\rangle$ on the cilinder states $|\,\xi(\varphi)\,\rangle$ in *configuration space*.

A. The Generalized Wigner function

Let us recall the Wigner function definition

$$W\,(q,p) \;=\; \int dv\; e^{(-pv/\hbar)}\;\; \Psi^*_\omega\left(q-\frac{v}{2}\right)\; \Psi_\omega\left(q+\frac{v}{2}\right)$$

where $(q,\, p)$ are the position and momenta as usual, or in general any canonical conjugate pair of variables. In our case $(m = 1 = r = \hbar)$, the position and momentum are (φ, j) :

$$(q,\, p) \;\to\; (\varphi, j)\,,$$

and expressing W in function of the complex variable z as before:

$$z = \omega\, e^{i\varphi},$$

we obtain the following generalized Wigner function:

$$W_{mn}\,(z,z^*) = \frac{1}{2\pi}\int d^2\eta\; \mathcal{M}\,(\,z_+\,) \sum_{m\,,\,n\,=\,0,1,2...}^{\infty} \frac{(\,z_+\,/2\,)^{2n}}{\sqrt{2n\,!}}\, \frac{\left(z^*_-\,/2\,\right)^{2m}}{\sqrt{2m\,!}} \mathcal{F}\,(\,z_+\,)\,\mathcal{F}\left(\,z^*_-\,\right)$$

where

$$z_{\pm} \equiv z \pm \eta / 2$$

$$\mathcal{M}(z+\eta/2) = \left(1 - |z + \eta/2|^2\right)^{1/2} \exp[-(z-z^*)(\eta+\eta^*)/2]$$

and

$$\mathcal{F}(z+\eta/2) = \left[1 + \left(1 - | z + \eta/2 |^2\right)^{1/2} \frac{(z + \eta/2)}{2\sqrt{2n+1}}\right]$$

$$\mathcal{F}(z^* - \eta^*/2) = \left[1 + \left(1 - | z + \eta/2 |^2\right)^{1/2} \frac{(z^* - \eta^*/2)}{2\sqrt{2n+1}}\right]$$

Notice that the complex variable is introduced in order to see the analytical conditions of the function in a little more detail.

Despite the degree of complexity of the function, we can provide an approximation in the case of $m = n$ (up to the leading terms inside the unitary disc e.g. $|\eta| < 1$):

$$W_{mm}(z, z^*) \approx 2\, e^{-4|z|^2}\left(2\,\mathrm{Ei}(4\,|z|^2) + \ln\frac{1}{|z|^4} + \cdots\right) \tag{22}$$

represented in Figure 3, Ei being the exponential integral function. This Wigner function for the circle states displays a classicalized typical shaped gaussian distribution, more bell-shaped that the square norm function of the states displayed in Figure 1.

VII. COSET COHERENT STATES FOR THE CIRCLE

A. Coset coherent states

Let us remind the definition of coset coherent states starting from a vector ϕ_0 invariant under the stability subgroup namely

$$H_0 = \{g \in G \mid \mathcal{U}(g)\,\phi_0 = \phi_0\} \subset G. \tag{23}$$

We can see that the orbit of ϕ_0 is isomorphic to the coset, e.g.

$$\mathcal{O}(\phi_0) \simeq G/H_0. \tag{24}$$

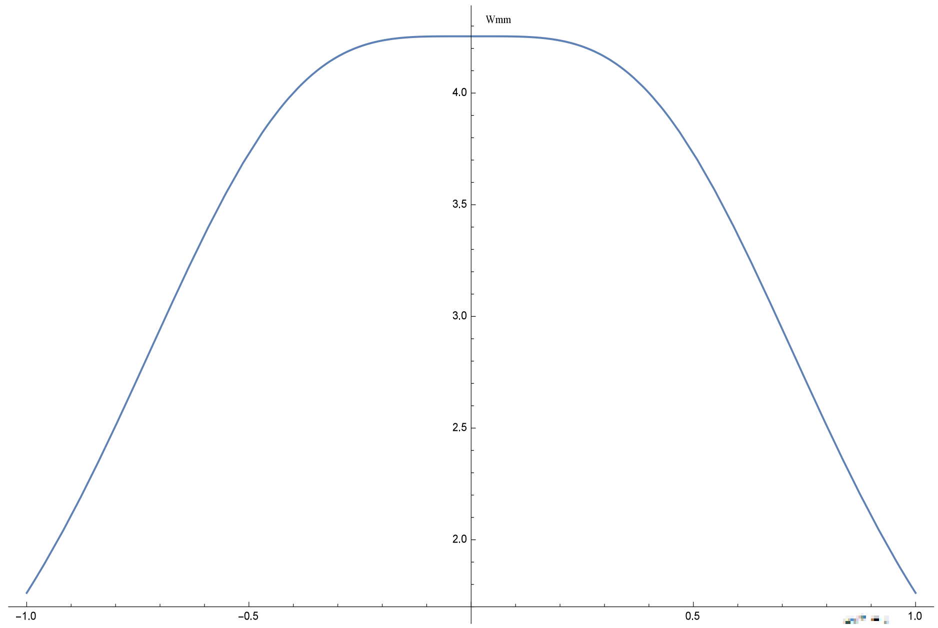

FIG. 3: Graphical representation of the generalized Wigner function for the approximate W_{mm} case: the shape of this distribution appears more bell-shaped than the function in Figure 1 (square norm).

On the other hand, if we remit to the operators, e.g.

$$|\phi_0\rangle \langle\phi_0| \equiv \widehat{\rho}_0 \tag{25}$$

then the orbit now is represented as

$$\mathcal{O}\left(\widehat{\rho}_0\right) \simeq G/H \tag{26}$$

with

$$\begin{aligned} H &= \{g \in G \mid \mathcal{U}(g)\,\phi_0 = \theta\phi_0\} \\ &= \left\{g \in G \mid \mathcal{U}(g)\,\widehat{\rho}_0\,\mathcal{U}^{\dagger}(g) = \widehat{\rho}_0\right\} \subset G \end{aligned} \tag{27}$$

The orbits are identified with cosets spaces of G with respect to the corresponding stability subgroups H_0 and H, that in the second case is defined within a phase.

Quantum viewpoint: From the quantum viewpoint: $|\phi_0\rangle \in \mathcal{H}$ (the Hilbert space) and $\rho_0 \in \mathcal{F}$ (the Fock space) are V_0 normalized fiducial vectors (embedded unit sphere in $\mathcal{H}$, and its real dimension is less than or equal to the dimension of G.).

Coherent state: A generalized coherent state system is now defined as the collection of unit vectors $\phi(g)$ comprising the orbit $\mathcal{O}(\phi_0)$; thus, it brings together [$\mathcal{H}$, G, $\mathcal{U}(g)$ and ϕ_0] in a special way, namely:

$$\mathcal{U}(G/H_0)\,\phi_0 = \phi(g)$$

B. Geometry of the group and coset

As a first step in the construction of the coset coherent states on the circle we take an element of the Euclidean group $E(2)$ in a matrix representation derived from the exponentiation of the generators of the algebra times the representative parameters of the coordinates, namely:

$$E(2) = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi & x \\ \sin\varphi & \cos\varphi & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in \mathbb{R}^2,\ \varphi \in S^1 \right\}$$

As can be seen, the circle (geometrically and topologically) is perfectly described by the coset $G/H = E(2)/\mathbb{T}_2$ with $\mathbb{T}_2$ being the group of translations in the plane as a stability subgroup.

C. Maurer-Cartan forms and vector fields

Let us to consider

$$E(2)^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & -x\cos\varphi - y\sin\varphi \\ -\sin\varphi & \cos\varphi & x\sin\theta - y\cos\varphi \\ 0 & 0 & 1 \end{pmatrix}$$

Then we can obtain the Maurer-Cartan forms via pullback e.g.

$$E(2)^{-1}\ dE(2) = \omega^{\varphi} g_{\varphi} + \omega^{x} g_{x} + \omega^{y} g_{y}$$

where $(g_\varphi,\, g_x,\, g_y)$ are the generators of the respective algebra, namely:

$$g_\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad g_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad g_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, we have

$$\begin{aligned} \omega^\varphi &= d\varphi \\ \omega^x &= \cos\varphi\, dx + \sin\varphi\, dy \\ \omega^y &= -\sin\varphi\, dx + \cos\varphi\, dy \end{aligned}$$

with the Cartan structure equations showing geometrically the closing of the E(2) algebra, namely,

$$\begin{aligned} d\omega^\varphi &= 0 = \omega^x \wedge \omega^y = dx \wedge dy \\ d\omega^x &= \omega^\varphi \wedge \omega^y \\ d\omega^y &= \omega^x \wedge \omega^\varphi \end{aligned}$$

From the above results, the vector fields can be computed in the standard manner

$$\begin{aligned} e_\varphi &= \partial_\varphi \\ e_x &= \cos\varphi\, \partial_x + \sin\varphi\, \partial_y \\ e_y &= -\sin\varphi\, \partial_x + \cos\varphi\, \partial_y \end{aligned}$$

with the commutation relations

$$[e_\varphi,\, e_x] = e_y, \qquad [e_\varphi,\, e_y] = -e_x, \qquad [e_x,\, e_y] = 0$$

D. Coset coherent states

The steps to follow for the determination of the coset coherent states are the following:

(i) The Coset G/H identification $\rightarrow E(2)/\mathbb{T}_2$, being $\mathbb{T}_2$ the group of translations $\{g_x, g_y\} \in \mathbb{T}_2$.

(ii) The Fiducial vector determination: It is annihilated by all the generators h of the stability subgroup H and for instance, invariant under the action of H. We propose:

$$| A_0 \rangle = A(\varphi, x, y) \; |\varphi\rangle$$

where $|\varphi\rangle$ is the London (circle) state, that is expanded in the $|n\rangle$ state of the harmonic oscillator and

$$A(\varphi, x, y)_{(\pm)} = (\cos\varphi \pm \sin\varphi)\, x + (\mp\cos\varphi + \sin\varphi)\, y,$$

such that we can see:

$$(e_x + e_y)\, A(\varphi, x, y)_{(\pm)} = 0$$

(iii) The coherent state is defined as the action of an element of the coset group on the fiducial vector $|A_0\rangle$, consequently the coherent state (still unnormalized yet) takes the form:

$$e^{-\alpha\,\partial_\varphi}\,|A_0\rangle = \frac{1}{\sqrt{2\pi}}\,(A_+\cos\alpha + A_-\sin\alpha) \sum_{n=0,1,2..} e^{-\alpha\,\partial_\varphi} e^{-i\,\varphi\, n}\,|\, n\,\rangle \tag{28}$$

$$|\,\alpha,\varphi\rangle = \frac{1}{\sqrt{2\pi}}\mathcal{S}(\alpha,\varphi) \underbrace{\sum_{n=0,1,2..} e^{-i(\varphi-\alpha/2)n}\,|n\rangle}_{|\varphi-\alpha/2\rangle}$$

$$= \frac{1}{\sqrt{2\pi}}\,\mathcal{S}(\alpha,\varphi)\; |\varphi - \alpha/2\,\rangle \tag{29}$$

where $\alpha \in \mathbb{C}$ is an arbitrary complex parameter in the element of the coset, which must be adjusted after normalization, being

$$\mathcal{S}(\alpha,\varphi) \equiv (A_+\cos\alpha + A_-\sin\alpha)$$

Some observations with respect to the expression Eq.(28) that is the most general coherent state expression from the Klauder-Perelomov construction viewpoint are the following: Notice that the state can be normalized from the overlap taking the form:

$$\langle\beta,\varphi'|\,|\alpha,\varphi\rangle = \frac{1}{2\pi}\mathcal{S}(\beta^*,\varphi')\,\mathcal{S}(\alpha,\varphi) \sum_{m=0,1,2..}\sum_{n=0,1,2..} e^{i\,(\varphi'-\beta^*/2\,)\,m} e^{-i(\varphi-a/2\,)\,n}\,\langle m|\,|n\rangle$$

$$= \frac{1}{2\pi}\mathcal{S}(\beta^*,\varphi')\,\mathcal{S}(\alpha,\varphi) \sum_{n=0,1,2..} e^{-i\,[\varphi-\varphi'-(\alpha-\beta^*)/2]\,n}$$

$$= \frac{1}{2\pi}\;\frac{\mathcal{S}(\beta^*,\varphi')\;\mathcal{S}(\alpha,\varphi)}{1-e^{-i\,(\,\varphi-\varphi'-(\alpha-\beta^*)/2\,)}}$$

Then, the state is fully normalizable: $\varphi \to \varphi'$ Iff the parameter α have $\mathrm{Im}\,\alpha \neq 0$

$$| \, |\alpha, \varphi\rangle \, |^2 \; = \; \frac{1}{2\pi} \; \frac{\mathcal{S}\,(\alpha^*, \varphi) \;\; \mathcal{S}\,(\alpha, \varphi)}{1 - e^{-\, i(\,\alpha^* - \alpha)/2}} \tag{30}$$

where:

$$\mathcal{S}\,(\alpha^*, \varphi)\,\mathcal{S}\,(\alpha, \varphi) = \left(x^2 + y^2\right)\cosh\left(2\,\mathrm{Im}\,\alpha\right) - \left(x^2 \; - \; y^2\right)\sin 2\left(\mathrm{Re}\,\alpha - \varphi\right) + 2xy\cos 2\left(\mathrm{Re}\,\alpha - \varphi\right)$$

Consequently, it solves the problem of the London states that are overcomplete but clearly not normalizable when $\varphi \to \varphi'$:

$$\langle\varphi| \, |\varphi'\rangle \;\; = \;\; \frac{1}{2\pi} \sum_{n\,=\,0,1,2..} e^{i(\varphi - \varphi')n} \;\; = \;\; \frac{1}{2\pi}\frac{1}{1 - e^{-i(\varphi - \varphi')}} \tag{31}$$

We see explicitely from these expressions Eq.(30), Eq.(31) how the general coherent states on the circle $|\alpha, \varphi\rangle$ (with the coherent characteristic complex parameter α) solve the problem of the non normalizability of the known (London, 't Hooft) $|\,\varphi\rangle$ states in the circle.

From Eq.(30) the normalized state coherent state $\langle\varphi| \, |\varphi'\rangle$ is:

$$|\alpha, \varphi\rangle \;\; = \;\; \underbrace{\sqrt{1 - e^{-\,\mathrm{Im}\,\alpha}}\; e^{i\,\arg\mathcal{S}}}_{\mathcal{N}} \sum_{n\,=\,0,1,2..} e^{-i(\varphi - \alpha/2)\,n} \;\; |n\rangle \tag{32}$$

However, the identity is not resolved in a strict sense, but in a weak sense, always for $\mathrm{Im}\,\alpha > 0$:

$$\int_0^{2\pi} |\alpha, \varphi\rangle \, \langle\alpha, \varphi| \; d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\,=\,0,1,2..} e^{i(m-n)\varphi} e^{i(\alpha n - \alpha^* m)/2} \; d\varphi \sum_{m\,=\,0,1,2...} \; \sum_{n\,=\,0,1,2..} |m\rangle \, \langle n|$$

$$= \sum_{n\,=\,0,1,2...}^{\infty} e^{-n\,\mathrm{Im}\,\alpha} \, |n\rangle \, \langle n| = \begin{pmatrix} 1 & & & & \\ & e^{-\,\mathrm{Im}\,\alpha} & & & \\ & & e^{-2\,\mathrm{Im}\,\alpha} & & \\ & & & \ddots & \\ & & & & e^{-n\,\mathrm{Im}\,\alpha} \end{pmatrix}$$

and which clearly shows the role played by the complex characteristic coherent state parameter α.

VIII. ACTION OF THE MP(2) GROUP ON THE COSET COHERENT STATES IN THE CIRCLE

Again, et's look at the sector $s = 1/4$ of the Hilbert space spanned by the $Mp(2)$ coherent states (unnormalized) , the basic state is

$$\left|\Psi^{(+)}(\omega)\right\rangle = \left(1-|\omega|^2\right)^{1/4} \sum_{n=0,1,2..} \frac{(\omega/2)^{2n}}{\sqrt{2n!}}\,|\,2n\,\rangle$$

On the other hand:

$$\langle\alpha,\varphi| = \mathcal{N}^* \sum_{n=0,1,2..} e^{i(\varphi-\alpha^*/2)n}\,\langle\, n\,| \tag{33}$$

Then, with all the definitions above, an in principle excluding the normalization $\mathcal{N}$ we have

$$\langle\alpha,\varphi|\left|\Psi^{(+)}(\omega)\right\rangle = \frac{\left(1-|\omega|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{m=0,1,2..}\sum_{n=0,1,2..} \frac{(\omega/2)^{2n}}{\sqrt{2n!}} e^{i(\varphi-\alpha^*/2)m}\,\langle m|\,|2n\rangle \tag{34}$$

$$\begin{aligned}\langle\alpha,\varphi|\left|\Psi^{(+)}(\omega)\right\rangle &= \frac{\left(1-|\omega|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{\left(\omega e^{i(\varphi-\alpha^*/2)}/2\right)^{2n}}{\sqrt{2n!}} = \\ &= \frac{\left(1-|\omega|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z'/2)^{2n}}{\sqrt{2n!}}\end{aligned} \tag{35}$$

with

$$\omega\, e^{i(\varphi-\alpha^*/2)} = z\, e^{-i\,\alpha^*/2} \equiv z'$$

: the analytic function in the disc is now modified by the complex phase $(\varphi-\alpha^*/2)$.

Similarly, for the sector $s = 3/4$ of the $Mp(2)$ states we have:

$$\begin{aligned}\langle\alpha,\varphi|\left|\Psi^{(-)}(\omega)\right\rangle &= \frac{\left(1-|\omega|^2\right)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{\left(\omega e^{i(\varphi-\alpha^*/2)}/2\right)^{2n+1}}{\sqrt{(2n+1)!}} \\ &= \frac{\left(1-|\omega|^2\right)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z'/2)^{2n+1}}{\sqrt{(2n+1)!}}\end{aligned} \tag{36}$$

Notice that by taking the scalar product between the coset coherent state $\langle\alpha,\varphi|$ and the coherent states of $Mp(2)$, $\left|\Psi^{(-)}(\omega)\right\rangle$ we obtain two non-equivalent expansions in terms of analytical functions on the disk for the sectors of the minimal representations $s = 1/4, 3/4$, *even and odd n states*, respectively, in the eigenstaes $|\,n\,\rangle$ of the harmonic oscillator. Consequently $\left(\omega\, e^{i\,(\varphi\alpha^*/2)} \equiv z'\right)$:

$$\langle\, \alpha, \varphi|\,\big|\Psi^{(\pm)}\,(z')\,\big\rangle = \begin{cases} \left(1-|z'|^2\right)^{1/4} \sum_{n=0,1,2..} \frac{(z'/2)^{2n}}{\sqrt{2n!}} & (+)\text{: even states} \\ \\ \left(1-|z'|^2\right)^{3/4} \sum_{n=0,1,2..} \frac{(z'/2)^{2n+1}}{\sqrt{(2n+1)!}} & (\text{-})\text{: odd states} \end{cases} \tag{37a}$$

Therefore, for the total projected state, $\langle\, \alpha, \varphi\,|\,|\,\Psi\,(z')\,\rangle$:

$$\langle\, \varphi|\,|\,\Psi\,(z')\,\rangle \;=\; \langle\, \varphi|\,\big|\,\Psi^{(+)}\,(z')\,\big\rangle \;+\; \langle\, \varphi|\,\big|\,\Psi^{(-)}\,(z')\,\big\rangle\;, \tag{37b}$$

We have

$$\langle\, \alpha, \varphi|\,|\Psi\,(z')\,\rangle = \left(1-|z'|^2\right)^{1/4} \sum_{n=0,1,2..} \frac{(z'/2)^{2n}}{\sqrt{2n!}} \left[1+\left(1-|z'|^2\right)^{1/2} \frac{(z'/2)}{\sqrt{2n+1}}\right] \tag{38}$$

We now consider the following observations:

(i) The analiticity condition of the function $\langle\varphi|\,|\Psi\,(z')\rangle$ on the disk now constrained, taking into account: $|\,z'| \;=\; |z|\,e^{-\,\mathrm{Im}\,\alpha/2} \;<\; 1$ which occurs under the already accepted condition of arising from the normalization function.

(ii) The topology of the circle induced by the coset coherent state $\langle\alpha, \varphi|$ Eq. (32) not only modifies the phase of ω (e.g: $\omega\, e^{i\,(\varphi-\alpha^*/2)} = z'$) but also the ratio of the disc due the displacement generated by the action of the coset.

(iii) The norm square of Eq. (38) is easily calculated giving as a result the function:

$$\begin{aligned} |\,\langle\varphi|\,|\Psi\,(z')\rangle\,|^2 \;=\; &\left(1-|\,z'|^2\right)^{1/2} \cosh\left(\frac{|z'|^2}{2}\right) \;+\; \left(1-|z'|^2\right)^{3/2} \sinh\left(\frac{|z'|^2}{2}\right) \;+ \\ &+\; \left(1-|\,z'|^2\right)^{1/2} \mathrm{Re}\,(z') \sum_{n\,=\,0,1,2..} \frac{|z'/2|^{\,4n}}{2n\,!\;(2n+1)}, \quad z' \;=\; \omega\, e^{i\,(\varphi-\alpha^*/2)} \end{aligned}$$

with a decreasing tail as n increases, and showing the analicity, in this case in the disc $|z'| < 1$, with the same comments as in the items (i)-(ii) above.

IX. CONCLUDING REMARKS

In this paper relevant implications inherent to the description of quantum theory, in particular when it takes a classical aspect, were elucidated and discussed from the principle of minimum group representation. To this end the concept of classical quantum duality was considered demonstrating the non-existence of ontological or hidden variables in the reality of the physical scenario analyzed.

(1) The application of the Minimal Representation Group Principle (MGRP) to the London state (circle-phase states), e.g. the application of an element of the $MP(2)$ group on the London state, *immediately classicalizes* the physical scenario considered: the quantum dynamics in the circle takes on the classical character.

(2) The application of the MGRP to the London states *naturally* introduces the analytic functions through the action of the basic (coherent, $s = 1/4$, $3/4$) states of the Metaplectic group in the Bargmann representation. This is in contrast with the case of t' Hooft in Ref [4] where although similar functions and results in the circle are considered, the analyticity in the unit disk $|\, z\, | < 1$, is introduced differently.

(3) The analytic functions induced by the action of the basic states of the metaplectic group and the London states divide the projection of the Hilbert space on the disk into *even* and *odd* functions that in all cases (both for the square of the norm and for the Wigner function) remain analytic inside the unit disk, as well as for its analytic extension (by inversion). In the Figure (4) we can clearly seen this fact.

(4) The application of the MGRP to the coherent states of the quantum particle on the cylinder (configuration space) of Refs [19], [20], [21], and Ref [22] in the non-orientable case, *classicizes the system* in a similar way to the London circle phase states but the

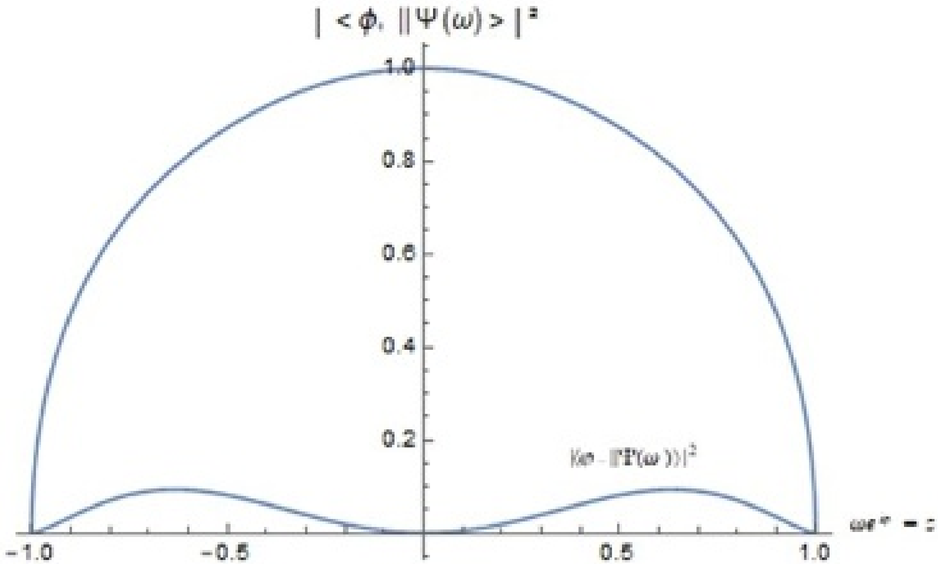

FIG. 4: In the Figure we see graphically represented the square norm of the projections under the application of the Minimal Group Representation, against the variable $z = \omega\, e^{i\,\varphi}$: The huge curve corresponds $|\langle\,\varphi_{+}|\,|\,\Psi\,(\omega)\,\rangle\,|^{2}$ to the $s = 1/4$, even n sector in the Hilbert space of the analytical functions, and the small curve $|\,\langle\varphi_{-}|\,|\Psi\,(\omega)\,\rangle\,|^{2}$ to the $s = 3/2$, odd n sector in the Hilbert space of the analytical functions).

square norm of the scalar product between the $Mp(2)$ states and the cilinder configuration coherent states has an extremely fast decay for increasing n due to the content of the factors $e^{-2\,n^{2}}$, $e^{-(\,2n+1/2\,)}$ and $e^{-(\,2n+1/2\,)^{2}}$, etc.

One of the reasons for the complexity of the obtained functions is due to the construction of the coherent states on the circle such as the non-standard modification of the Barut-Girardello definition for the Kowalski et al case, or the introduction of a Gaussian fiducial state for the Ref [21] proposal.

(5) In order to elucidate the classical-quantum duality problem by considering the circle topology in a complete way, *new coherent states for the circle* were introduced here. These coherent states follow Perelomov's definition Ref. [23] of coset coherent states, where the operators and the fiducial vector are completely determined by the nonlinear realization of the coset of the group E(2) on the group of translations in 2 dimensions as a stability group: $g = E(2)/\mathbb{T}_2$. These *new coherent states* solve the identity but in a weak way (diagonal matrix with entries $\mathbb{M}_{nn} = \mathrm{e}^{-n\,\mathrm{Im}\,\alpha}$), and the most important thing is that they are *completely normalizable*, well defined in the Hilbert space, contrary to the London states.

From all the cases exhaustively studied here $Mp(2)$ emerges as the *classical-quantum duality* group of symmetry .

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