

RESEARCH ARTICLE | JANUARY 09 2025

Classical (“ontological”) dual states in quantum theory and the minimal group representation Hilbert space

Diego J. Cirilo-Lombardo   ; Norma G. Sanchez 



APL Quantum 2, 016104 (2025)

<https://doi.org/10.1063/5.0247698>



Articles You May Be Interested In

Transforming underground to surface mining operation – A geotechnical perspective from case study

AIP Conference Proceedings (November 2021)

Monthly prediction of rainfall in nickel mine area with artificial neural network

AIP Conference Proceedings (November 2021)

Estimation of Karts groundwater based on geophysical methods in the Monggol Village, Saptosari District, Gunungkidul Regency

AIP Conference Proceedings (November 2021)



Special Topics Open for Submissions

[Learn More](#)

Classical (“ontological”) dual states in quantum theory and the minimal group representation Hilbert space

Cite as: APL Quantum 2, 016104 (2025); doi: 10.1063/5.0247698

Submitted: 8 November 2024 • Accepted: 11 December 2024 •

Published Online: 9 January 2025



View Online



Export Citation



CrossMark

Diego J. Cirilo-Lombardo^{1,2,a)}  and Norma G. Sanchez^{3,b)} 

AFFILIATIONS

¹M. V. Keldysh Institute of the Russian Academy of Sciences, Federal Research Center-Institute of Applied Mathematics, Miusskaya sq. 4, 125047 Moscow, Russian Federation

²CONICET-Universidad de Buenos Aires, Departamento de Física, Instituto de Física Interdisciplinaria y Aplicada (INFINA), Buenos Aires, Argentina

³The International School of Astrophysics Daniel Chalonge - Hector de Vega, CNRS, INSU-Institut National des Sciences de l'Univers, Sorbonne University, 75014 Paris, France

^{a)}Author to whom correspondence should be addressed: diego777jcl@gmail.com

^{b)}Norma.Sanchez@obspm.fr. URL: <https://chalonge-devega.fr/sanchez>

ABSTRACT

We investigate the classical aspects of quantum theory and under which description quantum theory does *appear classical*. Although such descriptions or variables are known as “ontological” or “hidden,” they are *not hidden* at all but are *dual classical* states (in the sense of the general classical–quantum duality of nature). We analyze and interpret the dynamical scenario in an inherent quantum structure: (i) We show that the use of the known $|\varphi\rangle$ states in the circle [F. London, *Z. Phys.* **37**, 915 (1926) and G. 't Hooft, “The hidden ontological variable in quantum harmonic oscillators,” arXiv 2407.18153 (2024)] takes a true dimension *only* when the system is subjected to the *minimal group representation* action of the metaplectic group $Mp(n)$. The $Mp(n)$ Hermitian structure fully covers the symplectic $Sp(n)$ group and, in certain cases, $OSp(n)$. (ii) We compare the circle $|\varphi\rangle$ states and the cylinder $|\xi\rangle$ states *in configuration space* with the two sectors of the full $Mp(2)$ Hilbert space corresponding to the *even* and *odd* n harmonic oscillators and their *total* sum. (iii) We compute the projections of the $Mp(2)$ states on the circle $\langle\varphi|$ and cylinder $\langle\xi|$ states. The known London circle states are *not normalizable*. We compute here the *general* coset coherent states $\langle\alpha, \varphi|$ in the circle, with α being the coherent complex parameter. It allows full *normalizability* of the complete set of the circle states. (iv) The London states (ontological in 't Hooft's description) *completely classicalize* the inherent quantum structure *only* under the action of the $Mp(n)$ *minimal group representation*. (v) For the coherent states in the cylinder (configuration space), all functions are analytic in the disk $|z = \omega e^{-i\varphi}| < 1$. For the general coset coherent states $|\alpha, \varphi\rangle$ in the circle, the complex variable is $z' = z e^{-i\alpha^*/2}$: The analytic function is modified by the complex phase $(\varphi - \alpha^*/2)$. (vi) The analyticity $|z'| = |z|e^{-\text{Im}\alpha/2} < 1$ occurs when $\text{Im}\alpha \neq 0$ because of normalizability and $\text{Im}\alpha > 0$ because of the identity condition. The circle topology induced by the $\langle\alpha, \varphi|$ coset coherent state also modifies the ratio of the disk due to the displacement by the coset. (vii) For the coset coherent cylinder states *in configuration space*, the classicalization is stronger due to screening exponential factors e^{-2n^2} , $e^{-(2n+1/2)}$, and $e^{-(2n+1/2)^2}$ for large n arising in the $Mp(2)$ projections on them. The generalized Wigner function shows a bell-shaped distribution and *stronger classicalization* than the square norm functions. The application of the minimal group representation immediately classicalizes the system, with $Mp(2)$ emerging as the group of the *classical–quantum duality* symmetry.

© 2025 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>). <https://doi.org/10.1063/5.0247698>

I. INTRODUCTION AND RESULTS

Nature is quantum. Classical states are contained in a quantum description. Quantum theory is in full development, whatever be in its own concepts and extensions, interpretations, and yet to be understood effects and manifestations, or in new applications, artificial intelligence, quantum computing, and technological and fundamental experimental research studies. See, for example, Refs. 1–3 and references therein. It is not the purpose of this introduction to review it here.

In this paper, without entering into all the issues of the interpretations of quantum theory, we consider the subject of the classical aspects of quantum theory and under which description quantum theory does appear classical.

Usually, the variables providing such a description are known as “ontological” or “local hidden variables,” although *they are not hidden at all*, and we agree with ’t Hooft (Ref. 4) who recently considered these variables as *not hidden*.

Usually, such classical or “ontological” variables are assumed to describe the results of the real measurements performed on a given quantum system (here, ontological means essential or existential, conceptual or substantial, for an entity or representation).

The general classical–quantum (wave–particle, de Broglie) duality is one deep foundational property of quantum theory and does remain a crucial one. More recently, this concept has been extended to include gravity at the Planck scale and beyond it (classical–quantum gravity duality), which is general and does not depend on the number or type of spacetime dimensions, manifolds (compactified or not), or other considerations.

The classical “ontological or existing” states are, in fact, *classical dual states*, and we do prefer this last term because it is physically precise, appropriate, and meaningful for these states.

The time evolution of the quantum harmonic oscillator does appear identical to the classical motion: a rigid rotating motion in phase space with the oscillator frequency. The quantum states of light are harmonic oscillators and have the same time evolution.

We also consider the *Wigner function*, which is a useful tool for the comparison of the classical and quantum dynamics of these states in phase space and analyzing the classical behavior in general. It allows us to obtain the probability distribution for these states in phase space. The Wigner or quasi-probability distribution, namely $W(q, p)$, is symmetric in the reflection (time and space) symmetries, e.g., $W(-q, -p) = W(q, p)$.

In this paper, we focus on the following:

- In order to interpret the dynamical scenario connected with an inherent quantum structure, we show that the use of the known states in the circle (’t Hooft 2024,⁴ and London 1926⁵) takes a true dimension only when the system is subjected to the minimal group representation under the action of the metaplectic group $Mp(n)$. Let us recall that $Mp(n)$ fully covers the symplectic group $Sp(n)$ and, in certain cases, its Hermitian structure can be extended to $OSp(n)$.
- We consider the metaplectic group $Mp(2n)$, e.g., the double covering of the symplectic group $Sp(2n)$. *Discretization* arises naturally and directly from the basic states of the metaplectic representations with an interesting feature to be highlighted here: The decomposition of the $SO(2, 1)$

group into its two irreducible representations encompasses the span of the *even* $|2n\rangle$ and *odd* $|2n+1\rangle$ states ($n = 1, 2, 3, \dots$) of the harmonic oscillator, whose entirety is covered by the metaplectic group. For $n \rightarrow \infty$, the spectrum becomes continuum as it must be.

- In the metaplectic representation, the general or complete states must be the sum of the two types of states, *even and odd* n states, because they span, respectively, the two sectors of the Hilbert space, $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, completely covered by the metaplectic symmetry group $Mp(2)$: $\mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$. Based on the highest eigenvalue of the number operator occurring in the complete Hilbert space, the two unitary irreducible representations of $Mp(2)$ are denoted as $\mathcal{H}_{1/4}$ (even states) and $\mathcal{H}_{3/4}$ (odd states).
- We compare the complete and fully normalizable $Mp(2)$ states with the $|\varphi\rangle$ states in the circle (London states, phase space, and the ontological states recently considered by ’t Hooft).
- We also consider general coherent states $|\xi\rangle$ in the cylinder *in configuration space* and compare them with the sectors of the $Mp(2)$ Hilbert space and with the *total* (sum of the two sectors) states (+) and (−), corresponding to the even ($2n$) and odd ($2n+1$) states of the harmonic oscillators. We compute the projections of the $Mp(2)$ $|\Psi^\pm(\omega)\rangle$ states on the circle $\langle\varphi|$ and cylinder $\langle\xi|$ states.
- The known circle (London, ’t Hooft) states are *coherent, non-normalizable states*. We compute here the *general* coherent states in the circle $\langle\alpha, \varphi|$, which include the complex parameter α characteristic of coherent states, whose meaning appears clearly allowing the fully *finite normalization* of the complete set of states. The power of the general *coset group construction procedure* of coherent states does show up here.

The main implications of the results of this paper are the following:

- (i) The London states (ontological in ’t Hooft’s description) *classicalize* completely the inherent quantum structure *only* under the application of the minimal group representation with the $Mp(n)$ group taking the main role.
- (ii) The action of the metaplectic group on the “ontological” (London) states breaks the invariance under time reversal assumed for the dynamics of the particle in the circle (arrow of time).
- (iii) In the case of the coherent states of a particle in the cylinder (configuration space), we can also assign to them the variable $z = \omega e^{-i\varphi}$ as in the case of the particle states in a circle (London states, phase space). All functions are analytic in the disk $|z| < 1$. For the general coherent states in the circle $|\alpha, \varphi\rangle$, the complex variable is $z' = \omega e^{i(\varphi - \alpha^*/2)} = z e^{-i\alpha^*/2}$: The analytic function in the disk is modified by the complex phase $(\varphi - \alpha^*/2)$, with α being the characteristic coherent state complex parameter.
- (iv) The analyticity condition on the disk $|z'| = |z|e^{-\text{Im}\alpha/2} < 1$ occurs with the condition $\text{Im}\alpha \neq 0$ arising from the normalizability and with $\text{Im}\alpha > 0$ arising from the identity condition. The topology of the circle

induced by the coset coherent state $\langle \alpha, \varphi |$ modifies not only the phase of ω (e.g., $\omega e^{i(\varphi - \alpha^*/2)} = z'$) but also the ratio of the disk due to the displacement generated by the action of the coset.

- (v) The norm of the projection of the $Mp(2)$ states on the cylinder and general circle coherent states clearly shows a very fast decrease as n increases or *classicalization*. In the case of the cylinder states in *configuration space*, the decrease is stronger than in the circle phase space states, due to the exponentials e^{-2n^2} , $e^{-(2n+1/2)}$, and $e^{-(2n+1/2)^2}$, arising in these configuration space projections.

The generalized Wigner function for the circle states displays a classicalized distribution, more bell-shaped than the square norm function of these states.

More remarks are presented in the Conclusions.

This paper is organized as follows: In Sec. II, we describe the quantum dynamics on the circle and its phase space states (London states). In Sec. III, we summarize the metaplectic minimal group representation approach, its group content, double covering, and fully complete Hilbert space of states. In Sec. IV, both the $Mp(2)$ basic states and the ontological states in the circle are compared with their mutual scalar products, which shows how classicalization occurs in this case. In Sec. V, we consider the general coherent states in a cylinder in configuration space and compute the projections of the $Mp(2)$ total states on them, which show an even stronger classicalization with respect to the London (circle, phase space) states. Section VI discusses the implications of the $Mp(2)$ representation reduction on these states and the analysis and results of Secs. II–V. In Secs. VII and VIII, the general normalized coset coherent states on the circle and the projected $Mp(2)$ reduction on them are computed and analyzed. Section IX summarizes the conclusions.

II. “ONTOLOGICAL” STATES AND THE MINIMAL GROUP REPRESENTATION

As we showed earlier in Ref. 6, there is an even more general principle in the fundamental structure of quantum spacetime: *the principle of minimal group representation*, which allows us to obtain, consistently and simultaneously, a natural description of the dynamics of spacetime and the physical states admissible within it.

The theoretical design is based on physical states, which are the mean values of the metaplectic group generators $Mp(n)$, the double covering $SL(2C)$ in vector representation, relative to the coherent states bearing the spin weight.

In this theoretical context, there is a connection between the dynamics given by the symmetry generators of the metaplectic group and the physical states (mappings of the generators through bilinear combinations of the ground states; Refs. 6 and 7).

Let us now see how to apply this principle to the problem of construction of classical variables in quantum theory as considered by ’t Hooft in Ref. 4. Therefore, we first need to consider the quantum dynamics of a particle on the circle, which we describe in Sec. II A.^{8–18}

A. Theoretical aspects of the quantum dynamics on the circle

- (i) The starting point to take into account to describe a free particle on a circle is its Hamiltonian,

$$H = L = \frac{1}{2} \dot{\varphi}^2(t) \quad (\text{e.g., } \dot{\varphi} = J),$$

with unit mass and velocity. Here, φ is the angle position, with a period of 2π ,

$$\varphi(t) = \varphi(0) + t.$$

Classically,

$$\{\varphi, J\} = 1.$$

Quantically (operator level),

$$[\widehat{\varphi}, \widehat{J}] = i. \tag{1}$$

The best candidate for the position operator (well defined in Hilbert space) is

$$U = e^{i\widehat{\varphi}} \quad (U \text{ is unitary}).$$

Moreover, it is easy to see from the above equations that

$$[\widehat{J}, U] = U. \tag{2}$$

- (ii) Let us now consider the eigenstates

$$\widehat{J}|j\rangle = j|j\rangle. \tag{3}$$

From Eqs. (2) and (3), we have

$$U|j\rangle = |j+1\rangle, \\ U^\dagger|j\rangle = |j-1\rangle,$$

where U^\dagger and U are ladder operators.

Now, it can be easily seen that the additional properties of the states $|j\rangle$ satisfy

$$\langle i | j \rangle = \delta_{ij} \quad (\text{orthogonality}),$$

$$\sum_j |j\rangle\langle j| = \mathbb{I} \quad (\text{completeness}).$$

As we see, we have all the ingredients to implement the principle of minimal representation of the group:

- (i) the basis $|j\rangle$ *isomorphic* to the basis $|n\rangle$ of the harmonic oscillator,
- (ii) a *symplectic structure*, and (iii) the commutator Eq. (1), that allows us to build the two operators

$$a = \frac{1}{\sqrt{2}}(\widehat{\varphi} + i\widehat{J}) \quad \text{and} \quad a^\dagger = \frac{1}{\sqrt{2}}(\widehat{\varphi} - i\widehat{J}),$$

where a and a^+ are ladder operators satisfying

$$[a, a^+] = 1. \tag{4}$$

Let us now consider the metaplectic group approach.

III. THE METAPLECTIC MINIMAL GROUP REPRESENTATION APPROACH

In Refs. 6 and 7 and references therein, a group-theoretic approach was developed to obtain the metric (line element) as the central geometrical object associated with a discrete quantum structure of the spacetime for a quantum theory of gravity.

In summary, for the purpose of this paper, the main characteristics of this framework are the following:

- Such an emergent metric is obtained from a Riemannian superspace and is described as a physical coherent state of the underlying cover of the $SL(2C)$ group: Interestingly, it seems necessary to consider the *full cover of the symplectic group*, which is the metaplectic group $Mp(n)$; its spectrum for all n leads, in particular for very large n , to continuous spacetime.
- The main and fundamental importance of this quantum description is based on the phase space of a relativistic particle in the *natural* superspace of bosonic and fermionic coordinates that allow preserving at the quantum level the square root forms of geometric operators (e.g., the Hamiltonian or the Lagrangian).
- Such a natural characteristic of this description is ensured by a complete bosonic realization of the underlying algebra through creation and annihilation operators of the harmonic oscillator that establish its gradation.
- The *discrete* structure of the spacetime arises directly from the basic states of the metaplectic representation, with an interesting feature to be highlighted here: The decomposition of the $SO(2,1)$ group into its two irreducible representations encompasses the span of *even* $|2n\rangle$ and *odd* $|2n+1\rangle$ states ($n = 1, 2, 3, \dots$), whose entirety is covered by the metaplectic group.
- In the metaplectic representation, the general or complete states must be the sum of the two types of states, the even and odd n states, because they span, respectively, the two sectors of the Hilbert space, $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$, whose complete covering is $\mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$ corresponding to the metaplectic symmetry group $Mp(2)$.
- This yields the relativistic quantum metric of the discrete structure spacetime as the fundamental basis for a quantum theory of gravity. For increasing numbers of levels n , the metric solution goes to the continuum and to the classical general relativistic manifold as it should be.
- The double covering of *even and odd* n states and *their sum* in order to have the complete Hilbert space reflects here the CPT completeness of the theory, and such a property is the reflection of unitarity. As we know, the metaplectic group $Mp(2)$ acts irreducibly on each of the subspaces $\mathcal{H}_{1/4}$ and $\mathcal{H}_{3/4}$ (even and odd sectors) by which the total Hilbert space

(i.e., \mathcal{H}) is divided, according to the $Mp(2)$ Casimir operator which gives precisely the values $k = 1/4, 3/4$,

$$K^2 = K_3^2 - K_1^2 - K_2^2 = k(k-1) = -\frac{3}{16} \mathbb{I}.$$

Therefore,

$$\mathcal{H}_{1/4} = \text{Span}\{|n \text{ even}\rangle \text{ states} : n = 0, 2, 4, 6, \dots\}, \tag{5}$$

$$\mathcal{H}_{3/4} = \text{Span}\{|n \text{ odd}\rangle \text{ states} : n = 1, 3, 5, 7, \dots\}. \tag{6}$$

Based on the highest eigenvalue of the number operator occurring in the complete $\mathcal{H} \equiv \mathcal{H}_{1/4} \oplus \mathcal{H}_{3/4}$,

$$T_3 |n\rangle = -\frac{1}{2} \left(n + \frac{1}{2} \right) |n\rangle.$$

The two unitary irreducible representations (UIRs) of $Mp(2)$ are denoted as

$$(UIR) \text{ restricted to } \mathcal{H}_{1/4} \rightarrow \mathcal{D}_{1/4} \in Mp(2), \tag{7}$$

$$(UIR) \text{ restricted to } \mathcal{H}_{3/4} \rightarrow \mathcal{D}_{3/4} \in Mp(2). \tag{8}$$

It is notable that in the general case, $Sp(2m)$ can be embedded somehow in a larger algebra as $(Sp(2m) + R^{2m})$, admitting a Hermitian structure with respect to which it becomes the orthosymplectic superalgebra $Osp(2m, 1)$.

Consequently, the metaplectic representation of $Sp(2m)$ extends to an irreducible representation (IR) of $Osp(2m, 1)$, which can be realized in terms of the space H_h of all holomorphic functions h ,

$$C^m \rightarrow C / \int |h(z)|^2 e^{-|z|^2} d\lambda(z) < \infty,$$

with $\lambda(z)$ being the Lebesgue measure on C^m .

The restriction of the $Mp(n)$ representation to $Sp(2m)$ implies that the two irreducible (even and odd n) sectors are supported by the subspaces H_h^\pm of the holomorphic function space H_h . H_h^+ and H_h^- are the (closed) spans of the set of functions

$$z^n \equiv (z_1^{n_1}, \dots, z_m^{n_m}),$$

where $n_\theta \in Z, |n| = \sum n_\theta$, *even* and *odd*, (H_h^+ and H_h^-), respectively.

A. $Mp(2)$, $SU(1,1)$, and $Sp(2)$

All the groups $Mp(2)$, $Sp(2, R)$, and $SU(1, 1)$ are three-dimensional. It is possible to parameterize them in several ways that make the homomorphic relations particularly simple. We use two of such parameterizations, in terms of the $Mp(2)$ group generators

(T_1, T_2, T_3) and the angles $(\alpha_1, \alpha_2, \alpha_3)$, both of which are described as

$$Mp(2) \rightarrow e^{-i\alpha_1 T_1}, e^{-i\alpha_2 T_2}, e^{-i\alpha_3 T_3},$$

$$Sp(2R) \rightarrow \begin{pmatrix} e^{\frac{1}{2}\alpha_1} & 0 \\ 0 & e^{-\frac{1}{2}\alpha_1} \end{pmatrix} \begin{pmatrix} \cosh \frac{1}{2}\alpha_2 & \sinh \frac{1}{2}\alpha_2 \\ \sinh \frac{1}{2}\alpha_2 & \cosh \frac{1}{2}\alpha_2 \end{pmatrix} \\ \times \begin{pmatrix} \cos \frac{1}{2}\alpha_3 & -\sin \frac{1}{2}\alpha_3 \\ \sin \frac{1}{2}\alpha_3 & \cos \frac{1}{2}\alpha_3 \end{pmatrix},$$

$$SU(1,1) \rightarrow \begin{pmatrix} \cosh \frac{1}{2}\alpha_1 & \sinh \frac{1}{2}\alpha_1 \\ \sinh \frac{1}{2}\alpha_1 & \cosh \frac{1}{2}\alpha_1 \end{pmatrix} \begin{pmatrix} \cosh \frac{1}{2}\alpha_2 & i \sinh \frac{1}{2}\alpha_2 \\ -i \sinh \frac{1}{2}\alpha_2 & \cosh \frac{1}{2}\alpha_2 \end{pmatrix} \\ \times \begin{pmatrix} e^{\frac{i}{2}\alpha_3} & 0 \\ 0 & e^{-\frac{i}{2}\alpha_3} \end{pmatrix},$$

where the angle α_3 has the range $(-4\pi, 4\pi]$ for $Mp(2)$ and the range $(-2\pi, 2\pi]$ for $Sp(2, R)$ and $SU(1, 1)$.

Let us consider the brief description of the relevant symmetry group to perform the realization of the Hamiltonian operator of the problem. This group specifically is the metaplectic $Mp(2)$ as well as the groups that are topologically covered by it. The generators of $Mp(2)$ are the following:

$$T_1 = \frac{1}{4} (qp + pq) = \frac{i}{4} (a^{+2} - a^2), \\ T_2 = \frac{1}{4} (p^2 - q^2) = -\frac{1}{4} (a^{+2} + a^2), \\ T_3 = -\frac{1}{4} (p^2 + q^2) = -\frac{1}{4} (a^+ a + a a^+), \quad (9)$$

with the commutation relations

$$[T_1, T_2] = -i T_3,$$

$$[T_3, T_1] = i T_2, \quad [T_3, T_2] = -i T_1,$$

with (q, p) , or alternatively (a, a^+) , being the variables of the standard harmonic oscillator, as usual.

If we rewrite the commutation relations as

$$[T_3, T_1 \pm i T_2] = \pm (T_1 \pm i T_2),$$

$$[T_1 + i T_2, T_1 - i T_2] = -2 T_3,$$

we see that the states $|n\rangle$ are eigenstates of T_3 such that

$$T_3 |n\rangle = -\frac{1}{2} \left(n + \frac{1}{2} \right) |n\rangle.$$

Moreover, it is easy to see that

$$T_1 + i T_2 = -\frac{i}{2} a^2, \quad T_1 - i T_2 = \frac{i}{2} a^{+2}.$$

IV. THE $Mp(2)$ BASIC STATES VS ONTOLOGICAL STATES IN THE CIRCLE

Let us look at the sectors $s = 1/4$ and $s = 3/4$ of the Hilbert space spanned by the $Mp(2)$ coherent states $|\Psi^{(\pm)}(\omega)\rangle$, with ω being the frequency.

For the $s = 1/4$ sector, the basic state is

$$|\Psi^{(+)}(\omega)\rangle = (1 - |\omega|^2)^{1/4} \sum_{n=0,1,2,\dots} \frac{(\omega/2)^{2n}}{\sqrt{2n!}} |2n\rangle.$$

On the other hand, the ontological states in the circle (in the limit $N \rightarrow 1$) considered by 't Hooft and previously proposed by London as phase states in Ref. 5 in terms of the eigenstates n of the harmonic oscillator are

$$\langle \varphi | = \frac{1}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} e^{i\varphi n} \langle n|. \quad (10)$$

They are overcomplete,

$$\langle \varphi | \varphi' \rangle = \frac{1}{2\pi} \sum_{m=0,1,2,\dots} \sum_{n=0,1,2,\dots} e^{i\varphi n} e^{-i\varphi' m} \langle n | m \rangle \\ = \frac{1}{2\pi} \sum_{n=0,1,2,\dots} e^{i(\varphi - \varphi') n},$$

and solve the identity

$$\int_0^{2\pi} |\varphi\rangle \langle \varphi| d\varphi = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\varphi} d\varphi \sum_{m=0,1,2,\dots} \sum_{n=0,1,2,\dots} |m\rangle \langle n| \\ = \sum_{n=0,1,2,\dots} |n\rangle \langle n| = \mathbb{I}.$$

Therefore, the scalar product of the two sets of states is

$$\langle \varphi | \Psi^{(+)}(\omega)\rangle = \frac{(1 - |\omega|^2)^{1/4}}{\sqrt{2\pi}} \sum_{m=0,1,2,\dots} \\ \times \sum_{n=0,1,2,\dots} \frac{(\omega/2)^{2n}}{\sqrt{2n!}} e^{i\varphi m} \langle m | 2n \rangle, \quad (11)$$

$$\langle \varphi | \Psi^{(+)}(\omega)\rangle = \frac{(1 - |\omega|^2)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(\omega e^{i\varphi}/2)^{2n}}{\sqrt{2n!}} \quad (12)$$

$$= \frac{(1 - |z|^2)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z/2)^{2n}}{\sqrt{2n!}}, \quad z = \omega e^{i\varphi}, \quad (13)$$

with $z = \omega e^{i\varphi}$, the analytic function in the disk is modified by the phase without changing the consistency of the conformal map.

Similarly, for the sector $s = 3/4$,

$$\langle \varphi | \Psi^{(-)}(\omega)\rangle = \frac{(1 - |\omega|^2)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(\omega e^{i\varphi}/2)^{2n+1}}{\sqrt{(2n+1)!}} \quad (14)$$

$$= \frac{(1 - |z|^2)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z/2)^{2n+1}}{\sqrt{(2n+1)!}}. \quad (15)$$

Notice that by taking the scalar product between the ontological states $|\varphi\rangle$ and the states of $Mp(2)$, we obtain *two non-equivalent* expansions in terms of analytical functions on the disk for the sectors of the minimal representations $s = 1/4, 3/4$, corresponding to the *even* and *odd* n eigenstates of the harmonic oscillator.

Consequently ($\omega e^{i\varphi} = z$),

$$\langle\varphi|\Psi^{(\pm)}(z)\rangle = \begin{cases} \frac{(1-|z|^2)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z/2)^{2n}}{\sqrt{(2n)!}}, & (+) : \text{even states,} \\ \frac{(1-|z|^2)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z/2)^{2n+1}}{\sqrt{(2n+1)!}}, & (-) : \text{odd states.} \end{cases} \quad (16a)$$

Therefore, the total or complete projected state $\langle\varphi|\Psi(z)\rangle$ is given by

$$\begin{aligned} \langle\varphi|\Psi(z)\rangle &= \langle\varphi|\Psi^{(+)}(z)\rangle + \langle\varphi|\Psi^{(-)}(z)\rangle, \quad (17) \\ \langle\varphi|\Psi(z)\rangle &= \frac{(1-|z|^2)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z/2)^{2n}}{\sqrt{(2n)!}} \\ &\quad \times \left[1 + (1-|z|^2)^{1/2} \frac{(z/2)}{\sqrt{2n+1}} \right]. \quad (18) \end{aligned}$$

Now let us consider the following observations about the total expression $\langle\varphi|\Psi(z)\rangle$:

- (i) The function $\langle\varphi|\Psi(z)\rangle$ is analytic on the unit disk: $|z| = |z| < 1$.
- (ii) The topology of the circle induced by the state $\langle\varphi|$ [Eq. (10)] only modifies the phase of ω (e.g., $\omega e^{i\varphi} = z$) in the projection $\langle\varphi|\Psi(\omega)\rangle$ [Eq. (18)]. This is so because the topology of the circle with $R = 1$ coincides with that of the unit disk.

- (iii) The norm of Eq. (18) is obtained, giving as a result the function

$$\begin{aligned} |\langle\varphi|\Psi(z)\rangle|^2 &= \frac{(1-|z|^2)^{1/2}}{2\pi} \sum_{n=0,1,2,\dots} \frac{|z/2|^{4n}}{(2n)!} \\ &\quad \times \left[1 + (1-|z|^2) \frac{|z/2|^2}{2n+1} \right] \\ &= \frac{(1-|z|^2)^{1/2}}{2\pi} \sum_{n=0,1,2,\dots} \\ &\quad \times \left[\frac{|z/2|^{4n}}{(2n)!} + (1-|z|^2)^{3/2} \frac{|z/2|^{2(2n+1)}}{(2n+1)!} \right] \\ &= (1-|z|^2)^{1/2} \cosh(|z|^2/2) \\ &\quad + (1-|z|^2)^{3/2} \sinh(|z|^2/2), \end{aligned}$$

which is evidently analytic in the disk $|z| = |\omega| < 1$, graphically represented in Figs. 1 and 2.

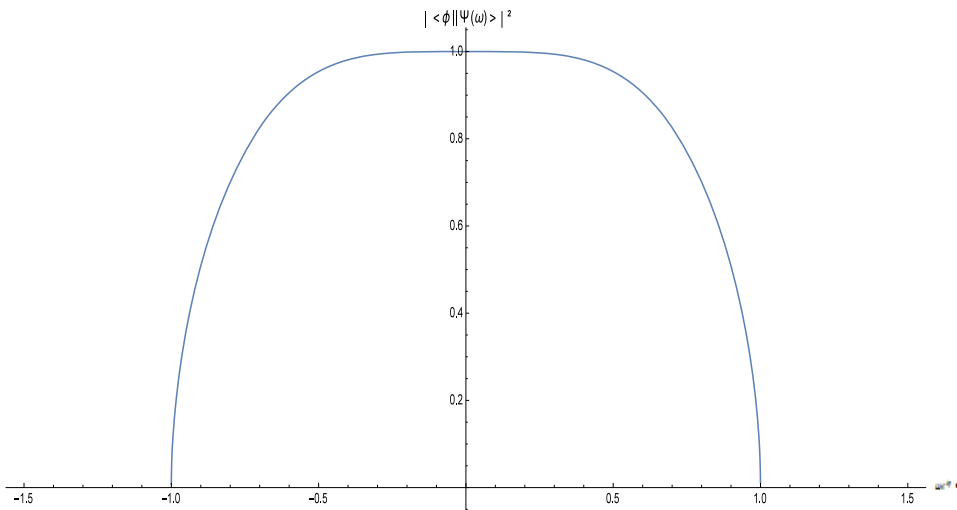


FIG. 1. Graphical representation of the norm of the projection of the total meta-plectic $Mp(2)$ states onto the circle $|\varphi\rangle$ states: the function $|\langle\varphi|\Psi(\omega)\rangle|^2$. As shown, analyticity is evident since it clearly respects $|z = \omega e^{i\varphi}| = |\omega| < 1$.

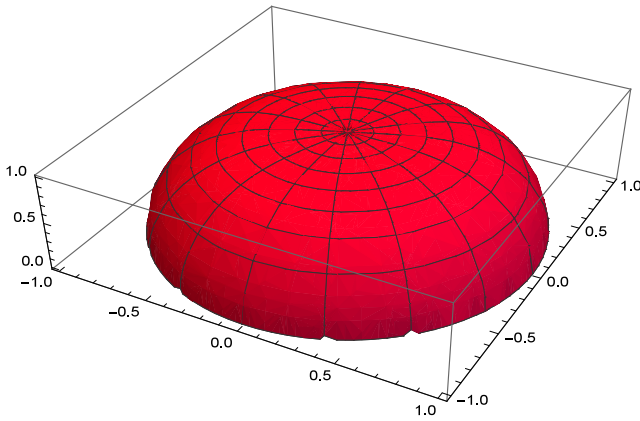


FIG. 2. Three-dimensional representation of the norm of the projection of the total metaplectic $Mp(2)$ states on the circle $|\varphi\rangle$ states: the function $|\langle\varphi|\Psi(\omega)\rangle|^2$, showing clearly the analytical character due to $|z = \omega e^{i\varphi}| = |\omega| < 1$.

V. GENERAL COHERENT STATES IN CONFIGURATION SPACE

In this section, we will elucidate and clarify the concept of the existence of an underlying quantum structure in physical systems, in light of the results of Sec. IV. For this purpose, we implement the minimal group representation approach to the case of the dynamics of a particle in the geometry of a cylinder. The dynamics of a particle on a cylinder have been studied in Refs. 19–22 for the non-orientable case.

In order to introduce the coherent states for a quantum particle on the cylinder geometry, it is possible to follow the Barut–Girardello construction and seek the coherent state as the solution of the eigenvalue equation,

$$X|\xi\rangle = \xi|\xi\rangle,$$

with ξ being the complex parameter, similar to the standard case, and

$$X = e^{i(\widehat{\varphi} + \widehat{T})}.$$

In order to analyze the coherent state of a particle in the cylinder in the context of the minimal group representation, we express the coherent states as

$$|\xi\rangle = \sum_{j=-\infty}^{\infty} e^{(l-i\varphi)j} e^{-j^2/2} |j\rangle. \quad (19)$$

If $|j\rangle \sim |n\rangle$, and for the $s = 1/4$ metaplectic states $|\Psi^{(+)}(\omega)\rangle$,

$$\begin{aligned} \langle\xi|\Psi^{(+)}(\omega)\rangle &= (1 - |\omega|^2)^{1/4} \sum_{m=-\infty}^{\infty} \sum_{n=0,1,2,\dots} \\ &\times \frac{(\omega/2)^{2n}}{\sqrt{2n!}} e^{(l-i\varphi)m} e^{-2m^2} \langle m|2n\rangle \\ &= (1 - |\omega|^2)^{1/4} \sum_{n=0,1,2,\dots} \frac{(\omega e^{(l-i\varphi)}/2)^{2n}}{\sqrt{2n!}} e^{-2n^2}. \end{aligned}$$

Similarly, for the $s = 3/4$ metaplectic states $|\Psi^{(-)}(\omega)\rangle$,

$$\begin{aligned} \langle\xi|\Psi^{(-)}(\omega)\rangle &= (1 - |\omega|^2)^{3/4} \sum_{m=-\infty}^{\infty} \sum_{n=0,1,2,\dots} \\ &\times \frac{(\omega/2)^{2n+1}}{\sqrt{(2n+1)!}} e^{(l-i\varphi)m} e^{-m^2/2} \langle m|2n+1\rangle \\ &= (1 - |\omega|^2)^{3/4} \sum_{n=0,1,2,\dots} \frac{(\omega e^{(l-i\varphi)}/2)^{2n+1}}{\sqrt{(2n+1)!}} e^{-(2n+1)^2/2}. \end{aligned}$$

We see that the scalar product projections taken with the cylinder $\langle\xi|$ space configuration states are similar to the projections taken with the circle $\langle\varphi|$ phase space states, but they contain the following weight functions: e^{-2n^2} and $e^{-(2n+1)^2/2}$, which *drastically attenuate* the scalar products when $n \rightarrow \infty$,

$$\langle\xi|\Psi^{(\pm)}(\omega)\rangle = \begin{cases} (1 - |\omega|^2)^{1/4} \sum_{n=0,1,2,\dots} \frac{(\omega e^{(l-i\varphi)}/2)^{2n}}{\sqrt{2n!}} e^{-2n^2}, & \text{even states,} \\ (1 - |\omega|^2)^{3/4} \sum_{n=0,1,2,\dots} \frac{(\omega e^{(l-i\varphi)}/2)^{2n+1}}{\sqrt{(2n+1)!}} e^{-(2n+1)^2/2}, & \text{odd states.} \end{cases} \quad (20a)$$

Consequently, for the total state

$$|\Psi(\omega)\rangle = |\Psi^{(+)}(\omega)\rangle + |\Psi^{(-)}(\omega)\rangle,$$

we have

$$\langle \xi | |\Psi(\omega)\rangle\rangle = (1 - |\omega|^2)^{1/4} \sum_{n=0,1,2 \dots} \frac{(\omega e^{(l-i\varphi)}/2)^{2n}}{\sqrt{2n!}} e^{-2n^2} \left[1 + (1 - |\omega|^2)^{1/2} \frac{\omega e^{(l-i\varphi)}}{\sqrt{2n+1}} e^{-(2n+1)/2} \right]. \tag{21}$$

We discuss the implications of these results in Sec. VI.

VI. IMPLICATIONS OF THE MINIMAL GROUP REPRESENTATION REDUCTION

As we have seen so far, in order to interpret the dynamical scenario connected with an inherent quantum structure, the use of the London circle states takes a true dimension only when the system is subjected to the minimal group representation under the action of the metaplectic group $Mp(n)$. Let us recall that $Mp(n)$ covers $Sp(n)$ twice and, in certain cases, its Hermitian structure can be extended to $Osp(n)$.

We outline some implications that result from the developments and analysis in Secs. II–V as follows:

- (i) The London states (ontological states in 't Hooft's description) *completely classicalize* the inherent quantum structure *only* under the application of the minimal group representation with the $Mp(n)$ group taking the main role.
- (ii) The action of the metaplectic group on the “ontological” (London) states breaks the invariance under time reversal assumed for the dynamics of the particle in the circle (arrow of time).
- (iii) In the case of the coherent states of a particle in the cylinder (configuration space) of Sec. V, we can set in our analysis the parameter $l = 0$. Thus, we can also assign to them the variable $z = \omega e^{-i\varphi}$ as in the case of the particle states in a circle (London states, phase space).
- (iv) In the case of the cylinder states, item (iii) with $l = 0$, the norm of the projection Eq. (21) is easily calculated giving as a result the function

$$|\langle \xi | |\Psi(\omega)\rangle\rangle|^2 = \sum_{n=0,1,2 \dots} \frac{|\omega/2|^{4n}}{2n!} e^{-2n^2} G(\omega, \varphi),$$

$$G(\omega, \varphi) = \left[1 + \frac{(1 - |\omega|^2)^{1/2}}{\sqrt{2n+1}} e^{-2n-1/2} \left(2\text{Re}(\omega e^{-i\varphi}) + e^{-2n-1/2} (1 - |\omega|^2)^{1/2} \frac{|\omega/2|^2}{\sqrt{2n+1}} \right) \right],$$

where we see the very fast decrease in the function due to the exponentials e^{-2n^2} , $e^{-(2n+1/2)}$, and $e^{-(2n+1/2)^2}$, arising in the projections of the metaplectic states $|\Psi(\omega)\rangle\rangle$ on the cylinder states $|\xi(\varphi)\rangle\rangle$ in *configuration space*.

A. The generalized Wigner function

Let us recall the Wigner function definition,

$$W(q, p) = \int dv e^{(-pv/\hbar)} \Psi_\omega^* \left(q - \frac{v}{2} \right) \Psi_\omega \left(q + \frac{v}{2} \right),$$

where (q, p) are the position and momentum as usual, or, in general, any canonical conjugate pair of variables. In our case ($m = 1 = r = \hbar$), the position and momentum are (φ, j) ,

$$(q, p) \rightarrow (\varphi, j),$$

and expressing W as a function of the complex variable z as before,

$$z = \omega e^{i\varphi},$$

we obtain the following generalized Wigner function:

$$W_{mn}(z, z^*) = \frac{1}{2\pi} \int d^2\eta \mathcal{M}(z_+) \sum_{m, n=0,1,2 \dots} \frac{(z_+/2)^{2n}}{\sqrt{2n!}} \frac{(z^*/2)^{2m}}{\sqrt{2m!}} \mathcal{F}(z_+) \mathcal{F}(z^*),$$

where

$$z_\pm \equiv z \pm \eta / 2,$$

$$\mathcal{M}(z + \eta/2) = (1 - |z + \eta/2|^2)^{1/2} \exp[-(z - z^*)(\eta + \eta^*)/2],$$

and

$$\mathcal{F}(z + \eta/2) = \left[1 + (1 - |z + \eta/2|^2)^{1/2} \frac{(z + \eta/2)}{2\sqrt{2n+1}} \right],$$

$$\mathcal{F}(z^* - \eta^*/2) = \left[1 + (1 - |z + \eta/2|^2)^{1/2} \frac{(z^* - \eta^*/2)}{2\sqrt{2n+1}} \right].$$

Notice that the complex variable is introduced in order to see the analytical conditions of the function in a little more detail.

Despite the degree of complexity of the function, we can provide an approximation in the case of $m = n$ (up to the leading terms inside the unitary disk, e.g., $|\eta| < 1$),

$$W_{mm}(z, z^*) \approx 2e^{-4|z|^2} \left(2 \text{Ei}(4|z|^2) + \ln \frac{1}{|z|^4} + \dots \right), \tag{22}$$

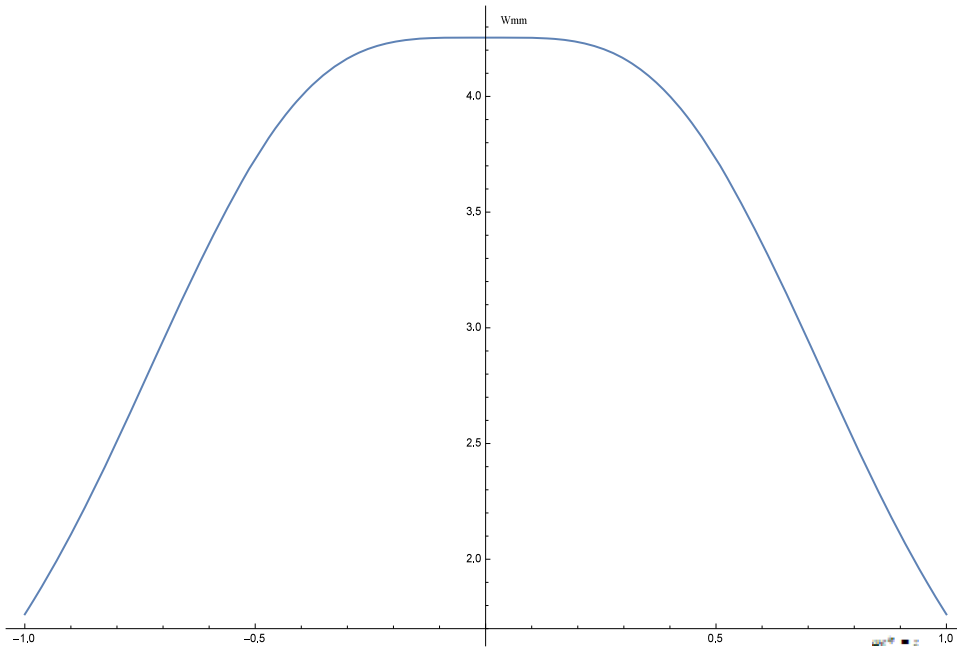


FIG. 3. Graphical representation of the generalized Wigner function for the approximate W_{mm} case: The shape of this distribution appears more bell-shaped than the function shown in Fig. 1 (square norm).

which is represented in Fig. 3, with E_i being the exponential integral function. This Wigner function for the circle states displays a classicalized typical shaped Gaussian distribution, more bell-shaped than the square norm function of the states displayed in Fig. 1.

VII. COSET COHERENT STATES FOR THE CIRCLE

A. Coset coherent states

Let us remind the definition of coset coherent states starting from a vector ϕ_0 invariant under the stability subgroup, namely

$$H_0 = \{g \in G | \mathcal{U}(g)\phi_0 = \phi_0\} \subset G. \tag{23}$$

We can see that the orbit of ϕ_0 is isomorphic to the coset, e.g.,

$$\mathcal{O}(\phi_0) \simeq G/H_0. \tag{24}$$

On the other hand, if we remit to the operators, e.g.,

$$|\phi_0\rangle\langle\phi_0| \equiv \widehat{\rho}_0, \tag{25}$$

then the orbit is now represented as

$$\mathcal{O}(\widehat{\rho}_0) \simeq G/H, \tag{26}$$

with

$$H = \{g \in G | \mathcal{U}(g)\phi_0 = \theta\phi_0\} = \{g \in G | \mathcal{U}(g)\widehat{\rho}_0 \mathcal{U}^\dagger(g) = \widehat{\rho}_0\} \subset G. \tag{27}$$

The orbits are identified with coset spaces of G with respect to the corresponding stability subgroups H_0 and H , which, in the second case, are defined within a phase.

Quantum viewpoint: From the quantum viewpoint, $|\phi_0\rangle \in \mathcal{H}$ (the Hilbert space) and $\rho_0 \in \mathcal{F}$ (the Fock space) are V_0 normalized fiducial vectors (embedded unit sphere in \mathcal{H} ; its real dimension is less than or equal to the dimension of G).

Coherent state: A generalized coherent state system is now defined as the collection of unit vectors $\phi(g)$ comprising the orbit $\mathcal{O}(\phi_0)$; thus, it brings together $[\mathcal{H}, G, \mathcal{U}(g), \text{ and } \phi_0]$ in a special way, namely

$$\mathcal{U}(G/H_0)\phi_0 = \phi(g).$$

B. Geometry of the group and coset

As a first step in the construction of the coset coherent states on the circle, we take an element of the Euclidean group $E(2)$ in a matrix representation derived from the exponentiation of the generators of the algebra times the representative parameters of the coordinates, namely

$$E(2) = \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi & x \\ \sin \varphi & \cos \varphi & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{R}^2, \varphi \in S^1 \right\}.$$

As can be seen, the circle (geometrically and topologically) is perfectly described by the coset $G/H = E(2)/\mathbb{T}_2$, with \mathbb{T}_2 being the group of translations in the plane as a stability subgroup.

C. Maurer–Cartan forms and vector fields

Let us consider

$$E(2)^{-1} = \begin{pmatrix} \cos \varphi & \sin \varphi & -x \cos \varphi - y \sin \varphi \\ -\sin \varphi & \cos \varphi & x \sin \varphi - y \cos \varphi \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, we can obtain the Maurer–Cartan form via pullback, e.g.,

$$E(2)^{-1} dE(2) = \omega^\varphi g_\varphi + \omega^x g_x + \omega^y g_y,$$

where (g_φ, g_x, g_y) are the generators of the respective algebra, namely

$$g_\varphi = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, we have

$$\begin{aligned} \omega^\varphi &= d\varphi, \\ \omega^x &= \cos \varphi dx + \sin \varphi dy, \\ \omega^y &= -\sin \varphi dx + \cos \varphi dy, \end{aligned}$$

with the Cartan structure equations showing geometrically the closing of the $E(2)$ algebra, namely

$$\begin{aligned} d\omega^\varphi &= 0 = \omega^x \wedge \omega^y = dx \wedge dy, \\ d\omega^x &= \omega^\varphi \wedge \omega^y, \\ d\omega^y &= \omega^x \wedge \omega^\varphi. \end{aligned}$$

From the above results, the vector fields can be computed in the standard manner,

$$\begin{aligned} e_\varphi &= \partial_\varphi, \\ e_x &= \cos \varphi \partial_x + \sin \varphi \partial_y, \\ e_y &= -\sin \varphi \partial_x + \cos \varphi \partial_y, \end{aligned}$$

with the commutation relations

$$[e_\varphi, e_x] = e_y, \quad [e_\varphi, e_y] = -e_x, \quad [e_x, e_y] = 0.$$

D. Coset coherent states

The steps to follow for the determination of the coset coherent states are the following:

- (i) The coset G/H identification $\rightarrow E(2)/\mathbb{T}_2$, with \mathbb{T}_2 being the group of translations $\{g_x, g_y\} \in \mathbb{T}_2$.
- (ii) The fiducial vector determination: It is annihilated by all the generators h of the stability subgroup H and, for instance, invariant under the action of H . We propose

$$|A_0\rangle = A(\varphi, x, y) |\varphi\rangle,$$

where $|\varphi\rangle$ is the London (circle) state, which is expanded in the $|n\rangle$ state of the harmonic oscillator and

$$A(\varphi, x, y)_{(\pm)} = (\cos \varphi \pm \sin \varphi)x + (\mp \cos \varphi + \sin \varphi)y,$$

such that we can see

$$(e_x + e_y)A(\varphi, x, y)_{(\pm)} = 0.$$

- (iii) The coherent state is defined as the action of an element of the coset group on the fiducial vector $|A_0\rangle$; consequently, the coherent state (still unnormalized yet) takes the form

$$e^{-\alpha \partial_\varphi} |A_0\rangle = \frac{1}{\sqrt{2\pi}} (A_+ \cos \alpha + A_- \sin \alpha) \sum_{n=0,1,2} e^{-\alpha \partial_\varphi} e^{-i\varphi n} |n\rangle, \tag{28}$$

$$\begin{aligned} |\alpha, \varphi\rangle &= \frac{1}{\sqrt{2\pi}} \mathcal{S}(\alpha, \varphi) \underbrace{\sum_{n=0,1,2} e^{-i(\varphi-\alpha/2)n} |n\rangle}_{|\varphi-\alpha/2\rangle} \\ &= \frac{1}{\sqrt{2\pi}} \mathcal{S}(\alpha, \varphi) |\varphi - \alpha/2\rangle, \end{aligned} \tag{29}$$

where $\alpha \in \mathbb{C}$ is an arbitrary complex parameter in the element of the coset, which must be adjusted after normalization as

$$\mathcal{S}(\alpha, \varphi) \equiv (A_+ \cos \alpha + A_- \sin \alpha).$$

Some observations with respect to Eq. (28), i.e., the most general coherent state expression from the Klauder–Perelomov construction viewpoint, are the following:

Notice that the state can be normalized from the overlap taking the form

$$\begin{aligned} \langle \beta, \varphi' | \alpha, \varphi \rangle &= \frac{1}{2\pi} \mathcal{S}(\beta^*, \varphi') \mathcal{S}(\alpha, \varphi) \sum_{m=0,1,2} \sum_{n=0,1,2} e^{i(\varphi' - \beta^*/2)m} e^{-i(\varphi - \alpha/2)n} \langle m | n \rangle \\ &= \frac{1}{2\pi} \mathcal{S}(\beta^*, \varphi') \mathcal{S}(\alpha, \varphi) \sum_{n=0,1,2} e^{-i[\varphi - \varphi' - (\alpha - \beta^*)/2]n} \\ &= \frac{1}{2\pi} \frac{\mathcal{S}(\beta^*, \varphi') \mathcal{S}(\alpha, \varphi)}{1 - e^{-i(\varphi - \varphi' - (\alpha - \beta^*)/2)}}. \end{aligned}$$

Then, the state is fully normalizable: $\varphi \rightarrow \varphi'$. If the parameter α has $\text{Im } \alpha \neq 0$,

$$\|\alpha, \varphi\rangle^2 = \frac{1}{2\pi} \frac{\mathcal{S}(\alpha^*, \varphi) \mathcal{S}(\alpha, \varphi)}{1 - e^{-i(\alpha^* - \alpha)/2}}, \tag{30}$$

where

$$\begin{aligned} \mathcal{S}(\alpha^*, \varphi) \mathcal{S}(\alpha, \varphi) &= (x^2 + y^2) \cosh(2 \text{Im } \alpha) - (x^2 - y^2) \\ &\quad \times \sin 2(\text{Re } \alpha - \varphi) + 2xy \cos 2(\text{Re } \alpha - \varphi). \end{aligned}$$

Consequently, it solves the problem of the London states that are overcomplete but clearly not normalizable when $\varphi \rightarrow \varphi'$,

$$\langle \varphi | \varphi' \rangle = \frac{1}{2\pi} \sum_{n=0,1,2} e^{i(\varphi - \varphi')n} = \frac{1}{2\pi} \frac{1}{1 - e^{-i(\varphi - \varphi')}}. \tag{31}$$

We see explicitly from Eqs. (30) and (31) how the general coherent states on the circle $|\alpha, \varphi\rangle$ (with the coherent characteristic complex parameter α) solve the problem of the non-normalizability of the known (London, 't Hooft) $|\varphi\rangle$ states in the circle.

From Eq. (30), the normalized coherent state $\langle\varphi|\varphi'\rangle$ is

$$|\alpha, \varphi\rangle = \underbrace{\sqrt{1 - e^{-\text{Im}\alpha}}}_{\mathcal{N}} e^{i \arg S} \sum_{n=0,1,2,\dots} e^{-i(\varphi-\alpha/2)n} |n\rangle. \quad (32)$$

However, the identity is not resolved in a strict sense, but in a weak sense, always for $\text{Im}\alpha > 0$,

$$\begin{aligned} \int_0^{2\pi} |\alpha, \varphi\rangle \langle\alpha, \varphi| d\varphi &= \frac{1}{2\pi} \int_0^{2\pi} \sum_{n=0,1,2,\dots} e^{i(m-n)\varphi} e^{i(\alpha n - \alpha^* m)/2} d\varphi \\ &\times \sum_{m=0,1,2,\dots} \sum_{n=0,1,2,\dots} |m\rangle \langle n| \\ &= \sum_{n=0,1,2,\dots} e^{-n \text{Im}\alpha} |n\rangle \langle n| \\ &= \begin{pmatrix} 1 & & & \\ & e^{-\text{Im}\alpha} & & \\ & & e^{-2 \text{Im}\alpha} & \\ & & & \ddots \\ & & & & e^{-n \text{Im}\alpha} \end{pmatrix}, \end{aligned}$$

which clearly shows the role played by the complex characteristic coherent state parameter α .

VIII. ACTION OF THE $MP(2)$ GROUP ON THE COSET COHERENT STATES IN THE CIRCLE

Again, let us look at the sector $s = 1/4$ of the Hilbert space spanned by the $Mp(2)$ coherent states (unnormalized). The basic state is given by

$$|\Psi^{(+)}(\omega)\rangle = (1 - |\omega|^2)^{1/4} \sum_{n=0,1,2,\dots} \frac{(\omega/2)^{2n}}{\sqrt{2n!}} |2n\rangle.$$

On the other hand,

$$\langle\alpha, \varphi| = \mathcal{N}^* \sum_{n=0,1,2,\dots} e^{i(\varphi-\alpha^*/2)n} \langle n|. \quad (33)$$

Then, with all the definitions given above and, in principle, excluding the normalization \mathcal{N} , we have

$$\begin{aligned} \langle\alpha, \varphi| \Psi^{(+)}(\omega)\rangle &= \frac{(1 - |\omega|^2)^{1/4}}{\sqrt{2\pi}} \sum_{m=0,1,2,\dots} \sum_{n=0,1,2,\dots} \\ &\times \frac{(\omega/2)^{2n}}{\sqrt{2n!}} e^{i(\varphi-\alpha^*/2)m} \langle m| |2n\rangle, \end{aligned} \quad (34)$$

$$\begin{aligned} \langle\alpha, \varphi| \Psi^{(+)}(\omega)\rangle &= \frac{(1 - |\omega|^2)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(\omega e^{i(\varphi-\alpha^*/2)})^{2n}}{\sqrt{2n!}} \\ &= \frac{(1 - |\omega|^2)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z'/2)^{2n}}{\sqrt{2n!}}, \end{aligned} \quad (35)$$

with

$$\omega e^{i(\varphi-\alpha^*/2)} = z e^{-i\alpha^*/2} \equiv z'.$$

The analytic function in the disk is now modified by the complex phase $(\varphi - \alpha^*/2)$.

Similarly, for the sector $s = 3/4$ of the $Mp(2)$ states, we have

$$\begin{aligned} \langle\alpha, \varphi| \Psi^{(-)}(\omega)\rangle &= \frac{(1 - |\omega|^2)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(\omega e^{i(\varphi-\alpha^*/2)})^{2n+1}}{\sqrt{(2n+1)!}} \\ &= \frac{(1 - |\omega|^2)^{3/4}}{\sqrt{2\pi}} \sum_{n=0,1,2,\dots} \frac{(z'/2)^{2n+1}}{\sqrt{(2n+1)!}}. \end{aligned} \quad (36)$$

Notice that by taking the scalar product between the coset coherent state $\langle\alpha, \varphi|$ and the coherent states of $Mp(2)$, $|\Psi^{(-)}(\omega)\rangle$, we obtain two non-equivalent expansions in terms of analytical functions on the disk for the sectors of the minimal representations $s = 1/4, 3/4$, even and odd n states, respectively, in the eigenstates $|n\rangle$ of the harmonic oscillator.

Consequently $(\omega e^{i(\varphi-\alpha^*/2)} \equiv z')$,

$$\langle\alpha, \varphi| \Psi^{(\pm)}(z')\rangle = \begin{cases} (1 - |z'|^2)^{1/4} \sum_{n=0,1,2,\dots} \frac{(z'/2)^{2n}}{\sqrt{2n!}}, & (+): \text{ even states,} \\ (1 - |z'|^2)^{3/4} \sum_{n=0,1,2,\dots} \frac{(z'/2)^{2n+1}}{\sqrt{(2n+1)!}}, & (-): \text{ odd states.} \end{cases} \quad (37a)$$

Therefore, for the total projected state, $\langle \alpha, \varphi | \Psi(z') \rangle$,

$$\langle \alpha, \varphi | \Psi(z') \rangle = \langle \alpha, \varphi | \Psi^{(+)}(z') \rangle + \langle \alpha, \varphi | \Psi^{(-)}(z') \rangle, \quad (37b)$$

we have

$$\langle \alpha, \varphi | \Psi(z') \rangle = (1 - |z'|^2)^{1/4} \sum_{n=0,1,2,\dots} \frac{(z'/2)^{2n}}{\sqrt{2n!}} \times \left[1 + (1 - |z'|^2)^{1/2} \frac{(z'/2)}{\sqrt{2n+1}} \right]. \quad (38)$$

We now consider the following observations:

- (i) The analyticity condition of the function $\langle \alpha, \varphi | \Psi(z') \rangle$ on the disk is now constrained, taking into account $|z'| = |z|e^{-\text{Im}\alpha/2} < 1$, which occurs under the already accepted condition of arising from the normalization function.
- (ii) The topology of the circle induced by the coset coherent state $\langle \alpha, \varphi |$ [Eq. (32)] modifies not only the phase of ω (e.g., $\omega e^{i(\varphi - \alpha^*/2)} = z'$) but also the ratio of the disk due to the displacement generated by the action of the coset.
- (iii) The norm square of Eq. (38) is easily calculated, giving as a result the function

$$|\langle \varphi | \Psi(z') \rangle|^2 = (1 - |z'|^2)^{1/2} \cosh\left(\frac{|z'|^2}{2}\right) + (1 - |z'|^2)^{3/2} \times \sinh\left(\frac{|z'|^2}{2}\right) + (1 - |z'|^2)^{1/2} \text{Re}(z') \sum_{n=0,1,2,\dots} \frac{|z'/2|^{4n}}{2n!(2n+1)}, \quad z' = \omega e^{i(\varphi - \alpha^*/2)},$$

with a decrease in the tail as n increases, showing the analyticity, in this case in the disk $|z'| < 1$, with the same comments as in items (i) and (ii) above.

IX. CONCLUDING REMARKS

In this paper, relevant implications inherent to the description of quantum theory, in particular when it takes a classical aspect, were elucidated and discussed from the principle of minimal group representation. To this end, the concept of classical quantum duality was considered demonstrating the non-existence of ontological or hidden variables in the reality of the physical scenario analyzed.

- (1) The application of the Minimal Representation Group Principle (MGRP) to the London state (circle, phase states), e.g., the application of an element of the $Mp(2)$ group on the London state, *immediately classicalizes* the physical scenario considered: The quantum dynamics in the circle takes on the classical character.
- (2) The application of the MGRP to the London states *naturally* introduces the analytic functions through the action of the basic (coherent, $s = 1/4, 3/4$) states of the metaplectic group in the Bargmann representation. This is in contrast to the

case of 't Hooft in Ref. 4, where although similar functions and results in the circle are considered, the analyticity in the unit disk $|z| < 1$ is introduced differently.

- (3) The analytic functions induced by the action of the basic states of the metaplectic group and the London states divide the projection of the Hilbert space on the disk into *even* and *odd* functions that, in all cases (both for the square of the norm and for the Wigner function), remain analytic inside the unit disk, as well as for its analytic extension (by inversion). In Fig. 4, we can clearly see this fact.
- (4) The application of the MGRP to the coherent states of the quantum particle on the cylinder (configuration space) of Refs. 19–22 in the non-orientable case *classicalizes the system* in a similar manner to the London circle phase states, but the square norm of the scalar product between the $Mp(2)$ states and the cylinder configuration coherent states has an extremely fast decay for increasing n due to the content of the factors e^{-2n^2} , $e^{-(2n+1/2)}$, and $e^{-(2n+1/2)^2}$.
- One of the reasons for the complexity of the obtained functions is due to the construction of the coherent states on the circle, such as the non-standard modification of the Barut–Girardello definition for the Kowalski *et al.* case or the introduction of a Gaussian fiducial state for the proposal in Ref. 21.
- (5) In order to elucidate the classical–quantum duality problem by considering the circle topology in a complete way, *new coherent states for the circle* were introduced here. These coherent states follow Perelomov’s definition (Ref. 23) of coset coherent states, where the operators and the fiducial vector are completely determined by the nonlinear realization of the coset of the group $E(2)$ on the group of translations in two dimensions as a stability group: $g = E(2)/\mathbb{T}_2$. These *new coherent states* solve the identity but in a weak way (diagonal matrix with entries $\mathbb{M}_{mn} = e^{-n \text{Im}\alpha}$), and the most important thing is that they are *completely normalizable*

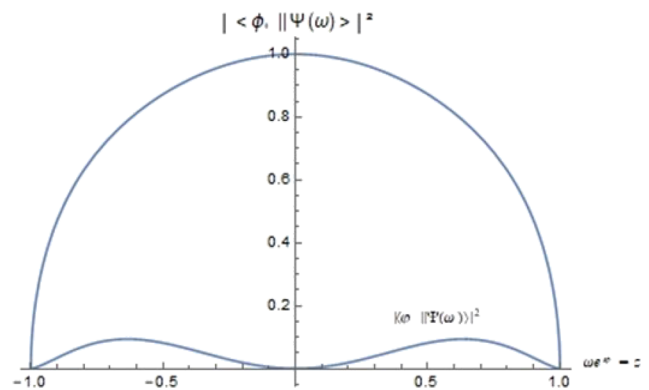


FIG. 4. Graphical representation of the square norm of the projections under the application of the minimal group representation, against the variable $z = \omega e^{i\varphi}$. The huge curve corresponds $|\langle \varphi_+ | \Psi(\omega) \rangle|^2$ to $s = 1/4$, even n sector in the Hilbert space of the analytical functions, and the small curve $|\langle \varphi_- | \Psi(\omega) \rangle|^2$ corresponds to $s = 3/2$, odd n sector in the Hilbert space of the analytical functions.

and well defined in the Hilbert space, contrary to the London states.

From all the cases exhaustively studied here, $Mp(2)$ emerges as the *classical–quantum duality* group of symmetry.

ACKNOWLEDGMENTS

D.J.C.-L. acknowledges CONICET and the Keldysh Institute of Applied Mathematics and the support of the Moscow Center of Fundamental and Applied Mathematics, in agreement with the Ministry of Science and Higher Education of the Russian Federation, No. 075-15-2022-283. N.G.S. acknowledges useful communications with Gerard 't Hooft and José Luis Mac Loughlin on various occasions.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Diego J. Cirilo-Lombardo: Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal). **Norma G. Sanchez:** Conceptualization (equal); Data curation (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Project administration (equal); Resources (equal); Software (equal); Supervision (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

- 1 A. Aspect, J. F. Clauser, and A. Zeilinger, Nobel Prize in Physics 2022, for experiments with entangled photons, . . . and pioneering quantum information science, <https://www.nobelprize.org/uploads/2023/10/advanced-physicsprize2022-4.pdf> and references therein.
- 2 J. L. Mac Loughlin and N. G. Sanchez, “The photography as a new conceptual quantum information system,” 2023, <https://www.researchgate.net/publication/375745724>, <https://www.bibsonomy.org/bibtex/7a77b52e38f27089f47229b1193dc005>; D. J. Cirilo-Lombardo, and N. G. Sanchez, “Entanglement and generalized Berry geometrical phases in quantum gravity,” *Symmetry* **16**(8), 1026 (2024).
- 3 J. J. Hopfield and G. E. Hinton, Nobel Prize in Physics 2024, for foundational discoveries and inventions that enable machine learning with artificial neural networks, <https://www.nobelprize.org/uploads/2024/10/advanced-physicsprize2024-2.pdf>, and references therein.
- 4 G. 't Hooft, “The hidden ontological variable in quantum harmonic oscillators,” *Front. Quantum Sci. Technol.* **3**, 1 (2024).
- 5 F. London, “Winkelvariable und Kanonische Transformationen in der Undulationsmechanik: Zeitschrift für Physik,” *Z. Phys.* **37**, 915–925 (1926).
- 6 D. J. Cirilo-Lombardo and N. G. Sanchez, *Universe* **10**, 22 (2024).
- 7 D. J. Cirilo-Lombardo and N. G. Sanchez, *Phys. Rev. D* **108**, 126001 (2023).
- 8 N. G. Sanchez, *Phys. Rev. D* **104**, 123517 (2021).
- 9 N. G. Sanchez, *Phys. Rev. D* **107**, 126018 (2023).
- 10 N. G. Sanchez, *Int. J. Mod. Phys. D* **28**, 1950055 (2019).
- 11 N. G. Sanchez, *Int. J. Mod. Phys. A* **34**, 1950155 (2019).
- 12 N. G. Sanchez, *Gravitation Cosmol.* **25**, 91–102 (2019).
- 13 D. J. Cirilo-Lombardo, *J. Math. Phys.* **57**(6), 063503 (2016).
- 14 D. J. Cirilo-Lombardo, *Found. Phys.* **37**, 919–950 (2007).
- 15 D. J. Cirilo-Lombardo, *Found. Phys.* **39**, 373–396 (2009).
- 16 J. R. Klauder and B. S. Skagerstam, *Coherent States: Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985).
- 17 J. R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics* (Dover Publications, 2006).
- 18 A. M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, Heidelberg, 1986).
- 19 K. Kowalski *et al.*, *J. Phys. A: Math. Gen.* **29**, 4149 (1996).
- 20 K. Kowalski and J. Rembielinski, *Phys. Rev. A* **75**, 052102 (2007).
- 21 J. A. Gonzalez and M. A. d. Olmo, *J. Phys. A: Math. Gen.* **31**, 8841 (1998).
- 22 D. J. Cirilo-Lombardo, *J. Phys. A: Math. Theor.* **45**, 244026 (2012).
- 23 A. M. Perelomov, “Coherent states for arbitrary Lie group,” *Commun. Math. Phys.* **26**, 222–236 (1972).