Theoretical Modeling of the Halo Mass Function: a Path-Integral Approach to the Excursion Set Theory

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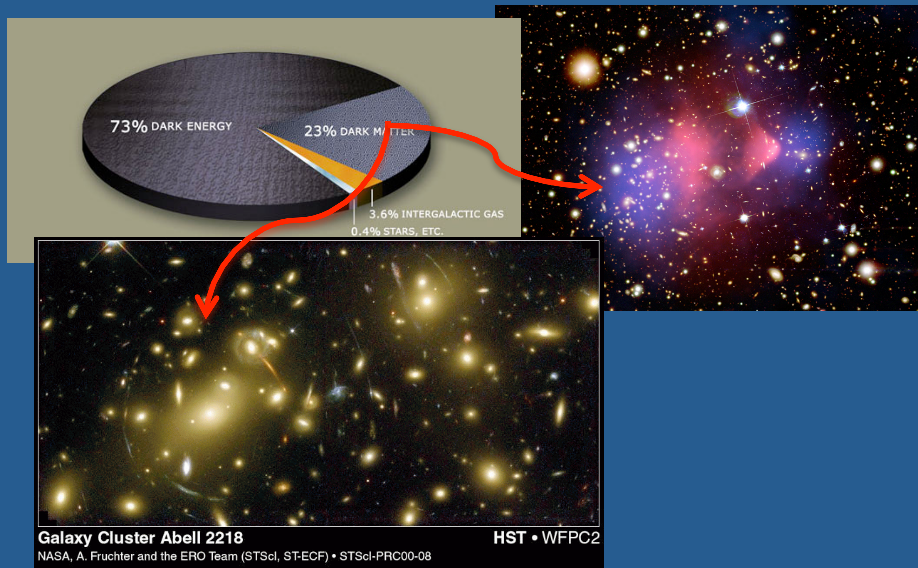
Cosmic Structures and Dark Matter

Dark Matter:

- Foster matter clustering

- Resides in virialized clumps



The Role of Halos:

- Building blocks of cosmic structure formation

- Shape baryon distribution

Halo Properties & Mass Function:

• Statistics of the initial density field

• Non-linear gravitational collapse

- Nature DM particles

- Underlying cosmology

Numerical N-body Simulations Results:

- Limited to Poisson-Vlasov systems (CDM)

- Still require theoretical understanding

$$f(\sigma) = A\left[\left(\frac{\sigma}{b}\right)^{-a} + 1\right] e^{-c/\sigma^2},$$

(Tinker et al. 2008)

Outline

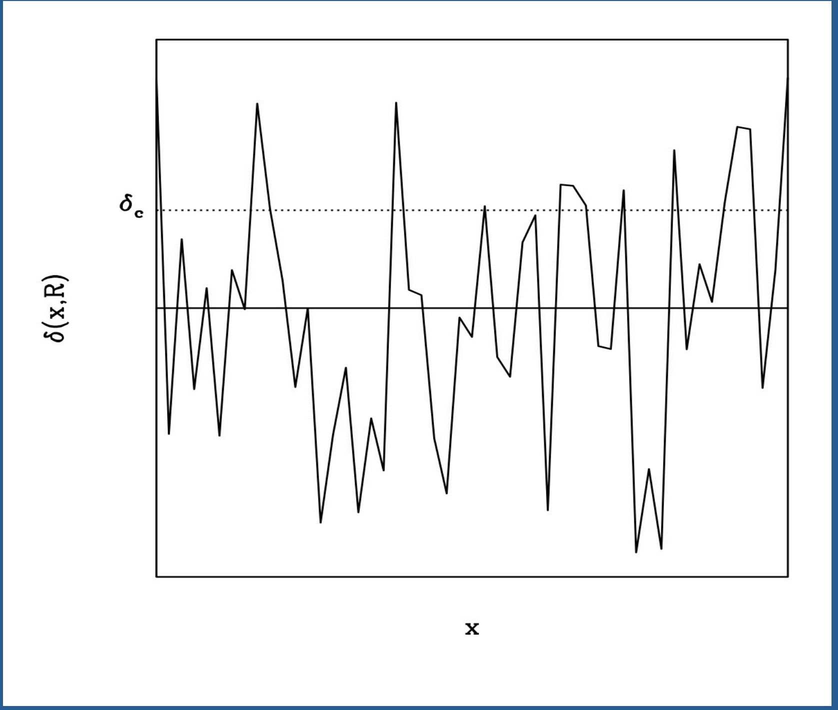
- Press-Schechter Formalism
- The Excursion Set Theory
- Halo Collapse Model
- Halo Mass Definition & Non-Markovianity
- Path-Integral Approach
- Comparison with N-body results

Reference List

- Bond et al., ApJ 379, 440 (1991)
- Percival, MNRAS 327, 1313 (2001)
- Maggiore & Riotto, ApJ 711, 907 (2010)
- Maggiore & Riotto, ApJ 717, 515 (2010)
- Maggiore & Riotto, ApJ 717, 526 (2010)
- De Simone, Maggiore & Riotto, MNRAS 412, 2587 (2011)
- Ma et al., MNRAS 411, 2644 (2011)
- Corasaniti & Achitouv, PRL in press (arXiv:1012.3468)
- Corasaniti & Achitouv, PRD submitted (on arXiv next week)
- Achitouv & Corasaniti, (on arXiv next week)

Press-Schechter Formalism

Smoothed Linear Density Field:



- Filtering scale and mass

$$M = \bar{\rho}\ V(R)$$

- Statistics of the smoothed density field $\Pi(\delta, R)$

- Linearly extrapolated collapse threshold δ_c

- #-halos with mass dM

$$\frac{dn}{dM} = \frac{1}{V}\frac{dF}{dM}$$

- Fraction of mass in halos > M

$$F_{PS}(R[M]) = \int_{\delta_c}^{\infty} d\delta\ \Pi(\delta, R[M])$$

Press-Schechter Mass Function

Gaussian Field:

- Variance of the smoothed field

$$\sigma^2(R) \equiv S(R) = \frac{1}{2\pi^2}\int k^2 P(k)\left|\tilde{W}(k,R)\right|^2 dk$$

- PDF

$$\Pi(\delta, S[R]) = \frac{1}{\sqrt{2\pi S[R]}} e^{-\delta^2/2S[R]}$$

- Spherical collapse threshold δ_c

- Fraction of mass in halos

$$F_{PS}(R) = \int_{\delta_c}^{\infty} d\delta\ \Pi(\delta, S(R)) = \frac{1}{2} Erfc\left[\frac{\delta_c}{\sigma(R)\sqrt{2}}\right]$$

- Halo Mass Function

$$\frac{dn}{dM} = \frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}}{d\ln M} f(\sigma)$$

with

$$f_{PS}(\sigma) = 2\sigma^2 \frac{dF_{PS}}{dS} = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma} e^{-\delta_c^2/(2\sigma^2)}$$

Cloud-in-Cloud Problem

Asymptotic Behavior:

- In the limit R → 0 all mass must be in collapsed structures, F(0)=1

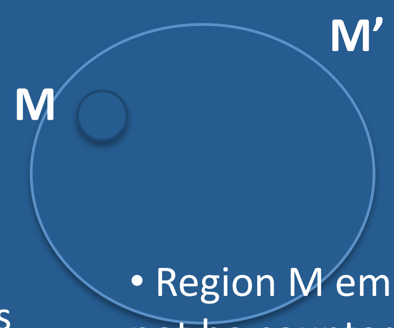
- In the PS calculation half of the mass is miscounted

$$F_{PS}(R) = \frac{1}{2} Erfc\left[\frac{\delta_c}{\sigma(R)\sqrt{2}}\right] \xrightarrow{R\to 0} \frac{1}{2}$$

Problem: In the PS approach there is no mass ordering

- No distinction between different configurations (cloud-in-cloud)

- Halo of mass M embedded in a larger non-collapsed region contributes to mass function at mass M



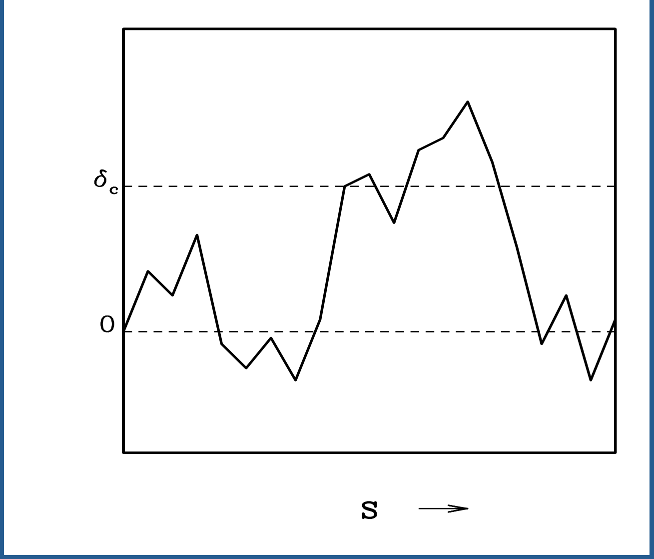
- Region M embedded in halo M'>M. M should not be counted in mass function since already included at M'

Excursion Set Theory

Key point:

$$\delta(x,R) = \frac{1}{(2\pi)^3}\int d^3k\ \delta(k)\tilde{W}(k,R)\ e^{-ikx}$$

- random walks start at R = ∞ (S=0) with δ = 0 evolving toward smaller R (larger S)



- At any point x, δ performs a random walk as function of R

- Langevin Equation:

$$\frac{\partial\delta}{\partial R} = \varsigma(R)\ \text{ and }\ \varsigma(R) = \frac{1}{(2\pi)^3}\int d^3k\ \delta(k)\frac{\partial\tilde{W}}{\partial R}e^{-ikx}$$

- ς(R) depends on Π(δ) and W(x,R)

- Halos of mass M corresponds to trajectories crossing the threshold at S(M)

- cloud-in-cloud solved by requiring first crossing

Excursion Set Mass Function

Stochastic Problem:

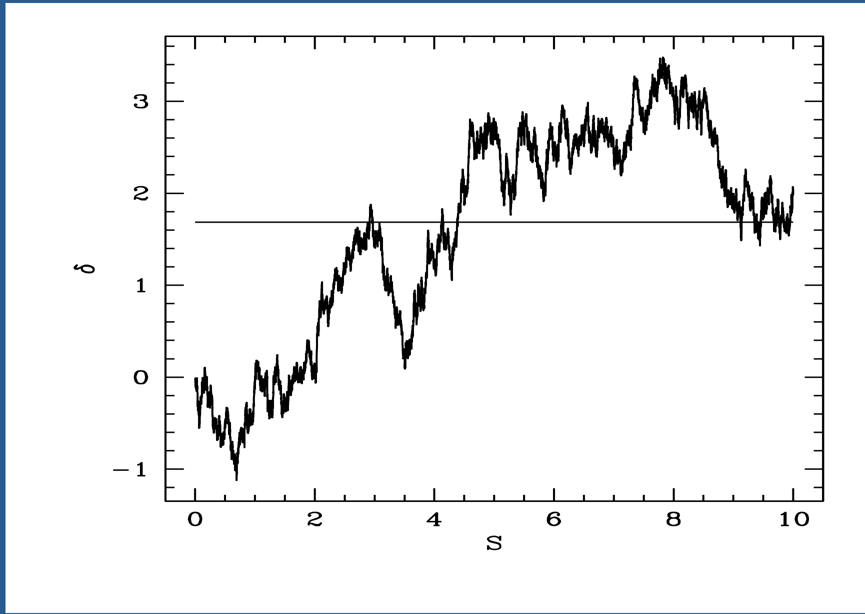
- Computation of the probability distribution of random walks with absorbing boundary, Π(δ,δ_c,S)
- Multiplicity function obtained from the first-crossing rate $\mathcal{F}$(S)=dF/dS

$$\int_0^S \mathcal{F}(S')\,dS' = 1 - \int_{-\infty}^{\delta_c} \Pi(\delta,\delta_c,S)\,d\delta \quad \Longrightarrow \quad \frac{dF}{dS} = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c} \Pi(\delta,\delta_c,S)\,d\delta$$

Sharp-k filter: $\tilde{W}(k,R) = \theta(1/R - k)$

- Markovian random walks

$$\frac{\partial \delta}{\partial S} = \eta(S) \quad \text{with} \quad \begin{aligned} &\langle \eta(S) \rangle = 0 \\ &\langle \eta(S)\eta(S') \rangle = \delta_D(S - S') \end{aligned}$$



Extended Press-Schechter

Fokker-Planck Equation:

$$\frac{\partial \Pi}{\partial S} = \frac{1}{2}\frac{\partial^2 \Pi}{\partial \delta^2}$$

with

$$\Pi(\delta,0) = \delta_D(\delta)$$
$$\Pi(\delta_c,S) = 0$$

$$\Pi(\delta,S) = \frac{1}{\sqrt{2\pi S}}\left[e^{-\delta^2/(2S)} - e^{-(2\delta_c-\delta)^2/(2S)}\right]$$

for $\delta < \delta_c$

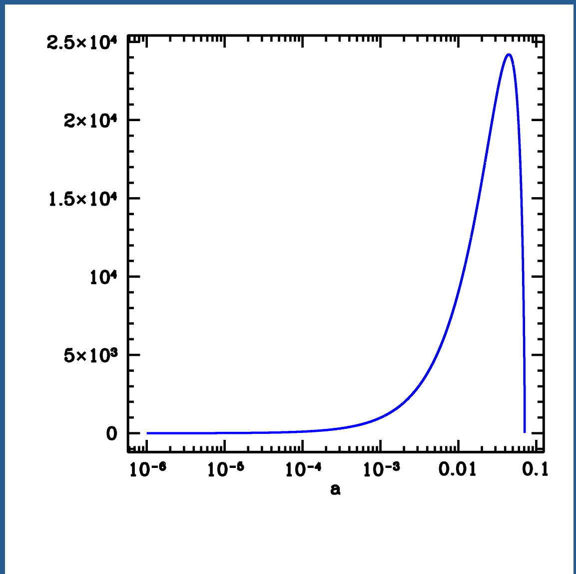
$$\frac{dF}{dS} = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c}\Pi(\delta,\delta_c,S)\,d\delta \longrightarrow f_{EPS}(\sigma) = \sqrt{\frac{2}{\pi}}\,\frac{\delta_c}{\sigma}\,e^{-\delta_c^2/(2\sigma^2)}$$

Halo Collapse Model

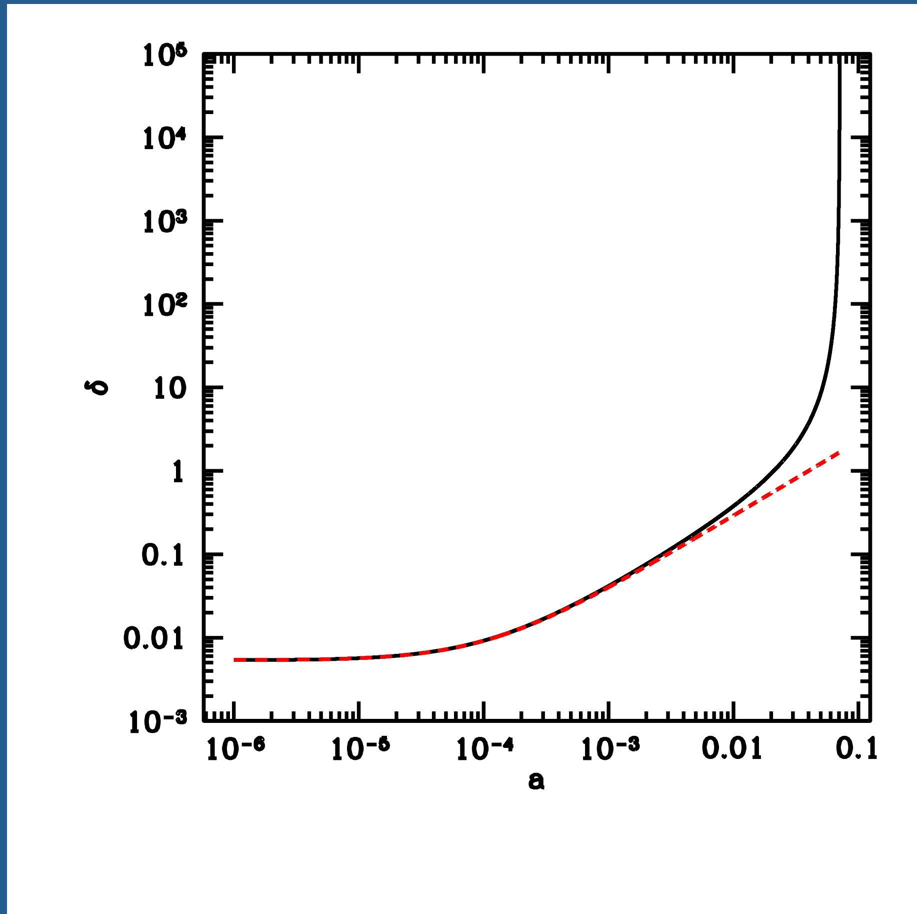
Spherical Collapse (Gunn & Gott, 1973)

- Top-hat perturbation in FRW background, $y=R/R_i$
- Dynamics independent of R_i

$$\ddot{y} = -\frac{4}{3}\pi G \bar{\rho}_m^i (1+\delta_m^i)\frac{1}{y^2}$$



$$1+\delta_m = (1+\delta_m^i)y^3\left(\frac{a}{a_i}\right)^3$$



Non-Spherical Halo Collapse

Ellipsoidal Collapse

- Initial Gaussian fluctuations are non-spherical (Doroshkevich, 1970)
- Ellipsoidal halos collapse and shear

$$\frac{\mathrm{d}^2 a_i}{\mathrm{d}t^2} = \frac{8}{3}\pi G\bar{\rho}_\Lambda a_i - 4\pi G\bar{\rho}_m a_i\left[\frac{1}{3} + \frac{\Delta(t)}{3} + \frac{b_i'(t)}{2}\Delta(t) + \lambda_i'(t)\right]$$

$$b_i' = -\frac{2}{3} + a_1 a_2 a_3 \int_0^\infty \frac{\mathrm{d}\tau}{(a_i^2+\tau)\Pi_{m=1}^3 (a_m^2+\tau)^{1/2}}$$

$$\lambda_1 = \frac{\delta}{3}(1-3e+p)$$

$$\lambda_2 = \frac{\delta}{3}(1-2p)$$

$$\lambda_3 = \frac{\delta}{3}(1+3e+p)$$

- Dynamics dependent of initial size of the collapsing region
- Critical Overdensity is mass dependent (e.g. Eisenstein & Loeb, 1995)

"Fuzzy" Barrier

Stochastic Barrier

- Ellipsoidal parameters are random variables with characteristic probability distribution

- e.g. for a Gaussian density field

$$g(e,p|\delta) = \frac{1125}{\sqrt{10\pi}} e(e^2 - p^2)\left(\frac{\delta}{\sigma}\right)^5 \exp\left[-\frac{5}{2}\frac{\delta^2}{\sigma^2}(3e^2 + p^2)\right]$$

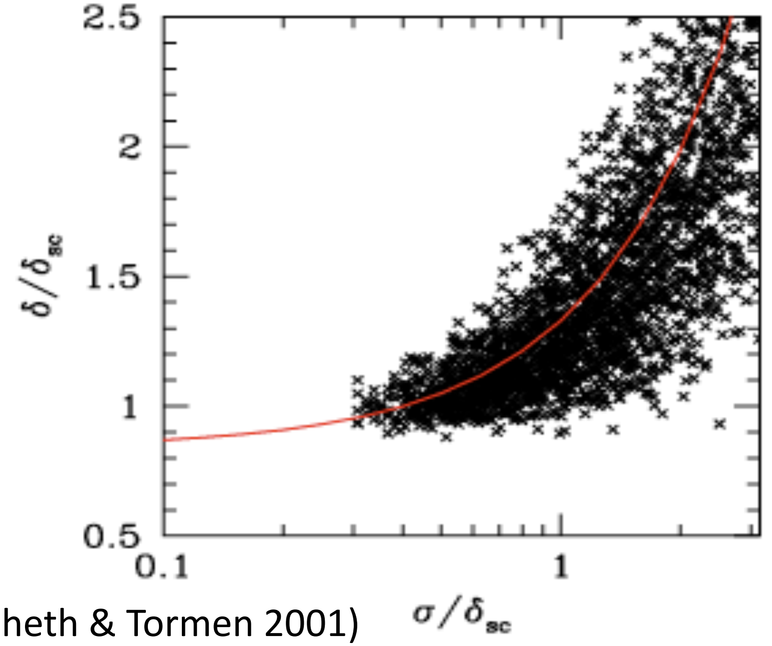
- Density threshold is a random variable

- e.g $\langle B(S)\rangle = \delta_c[1+\beta\,(S/S_*)^\gamma]$

(Sheth, Mo & Tormen 2001)

- Stochastic barrier model: specify the moments of barrier's PDF

$\langle B(S)\rangle$ & $\langle (B(S)-\langle B(S)\rangle)^2\rangle$, …



(Sheth & Tormen 2001)

Stochastic Barrier and Excursion Set

Non-Spherical Collapse

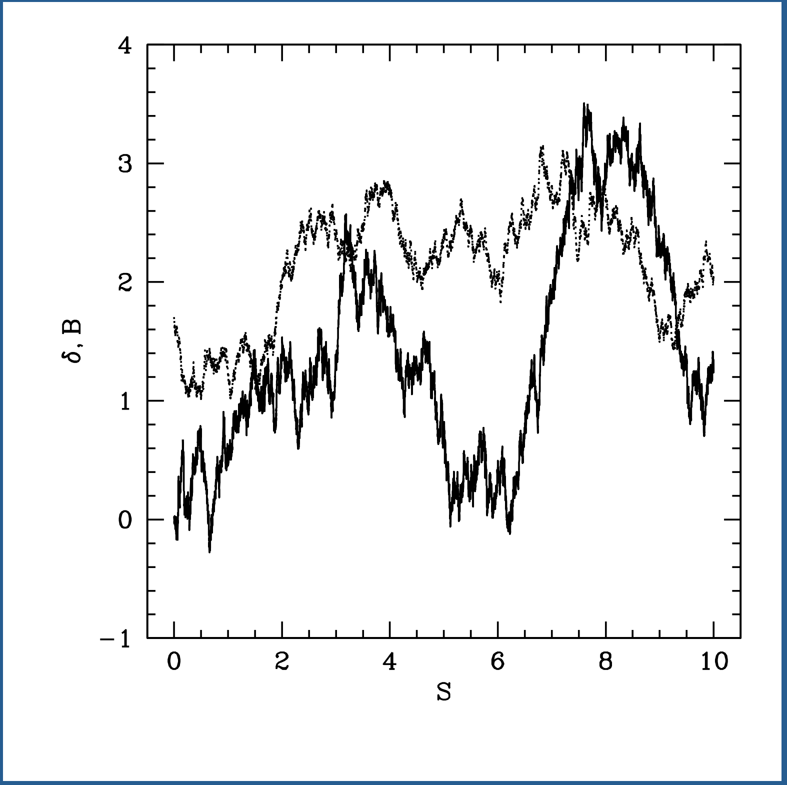
- Barrier performs a random walk characterized by non-spherical collapse predicted correlators
- e.g. $\langle B(S)\rangle = \delta_c + \beta S$ & $\langle (B(S) - \langle B(S)\rangle)^2\rangle^{1/2} = \sqrt{D_B}\,\sigma$
- Introduce: $Y = B - \delta$

Sharp-k filter case:

$$\frac{\partial Y}{\partial S} = \beta + \eta(S)$$

$$\langle \eta(S) \rangle = 0$$

$$\langle \eta(S)\eta(S') \rangle = (1 + D_B)\delta_D(S - S')$$



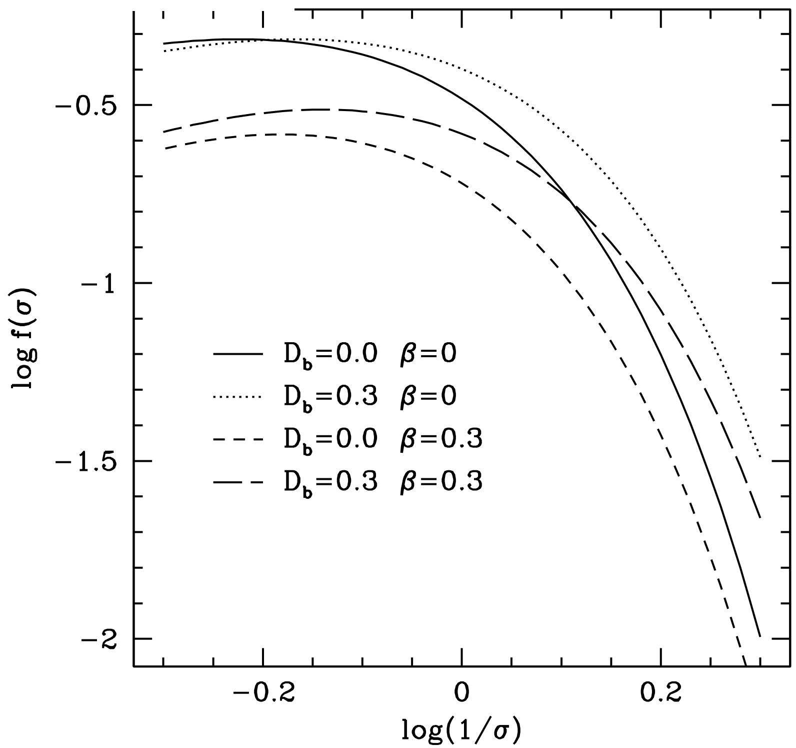
$$\frac{\partial \Pi}{\partial S} = -\beta \frac{\partial \Pi}{\partial Y} + \frac{1 + D_B}{2} \frac{\partial^2 \Pi}{\partial Y^2}$$

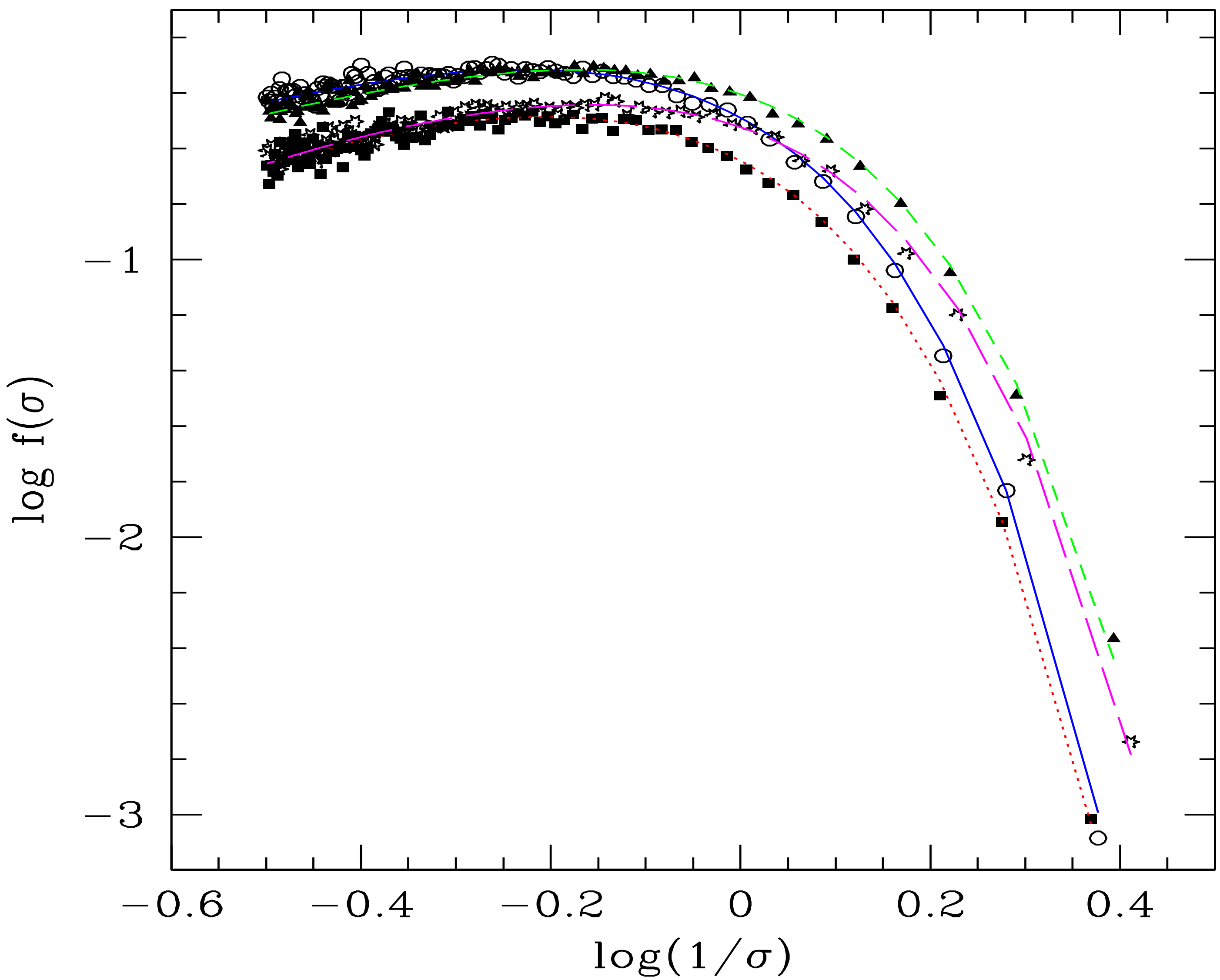
$$\Pi(Y,0) = \delta_D(Y - \delta_c)$$
$$\Pi(0,S) = 0$$

$$\Pi(Y,S) = \frac{e^{\frac{\beta}{1+D_B}(Y - Y_0 - \beta S/2)}}{\sqrt{2\pi S(1 + D_B)}} \left[e^{-\frac{(Y-Y_0)^2}{2S(1+D_B)}} - e^{-\frac{(Y+Y_0)^2}{2S(1+D_B)}} \right]$$

Multiplicity Function:

$$f(\sigma) = \sqrt{\frac{2}{\pi}} \frac{\delta_c}{\sigma\sqrt{1 + D_B}} e^{-\frac{(\delta_c + \beta \sigma^2)^2}{2\sigma^2 (1+D_B)}}$$





−1
−2
−3
log f(σ)
−0.6
−0.4
−0.2
0
0.2
0.4
log(1/σ)

Filter Function and Halo Mass Definition

Mass and smoothing scale: $M(R) = \rho \; V(R)$ with $V(R) = \int d^3x \; W(x,R)$

- Unambiguously define only for sharp-x filter: $W(x,R) = \theta(|\vec{x}| - R)$
- Generic filters define M up to a normalization constant
- Sharp-k leaves M undefined

N-body Halo mass:

- Mass depends on Halo detection algorithm
- e.g SOD $M_\Delta = \frac{4}{3}\pi R_\Delta^3 \bar{\rho} \; \Delta$ corresponding to $W(x,R_\Delta) = \theta(|\vec{x}| - R_\Delta)$

Realistic Filtering: Sharp-x

- Leads to correlated random walks since $\tilde{W}(k,R) = 3\frac{\sin(kR) - (kR)\cos(kR)}{(kR)^3}$

Top-Hat Filter and Correlated Random Walks

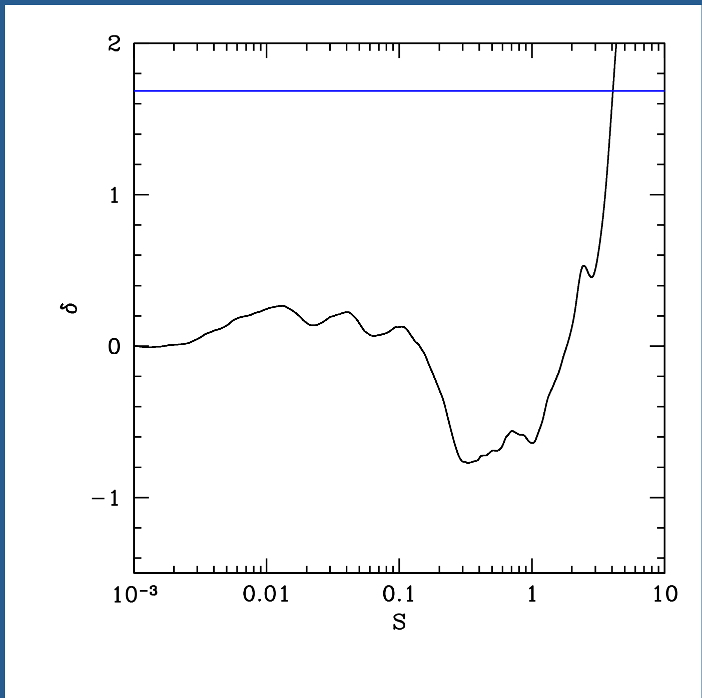
Langevin Equation:

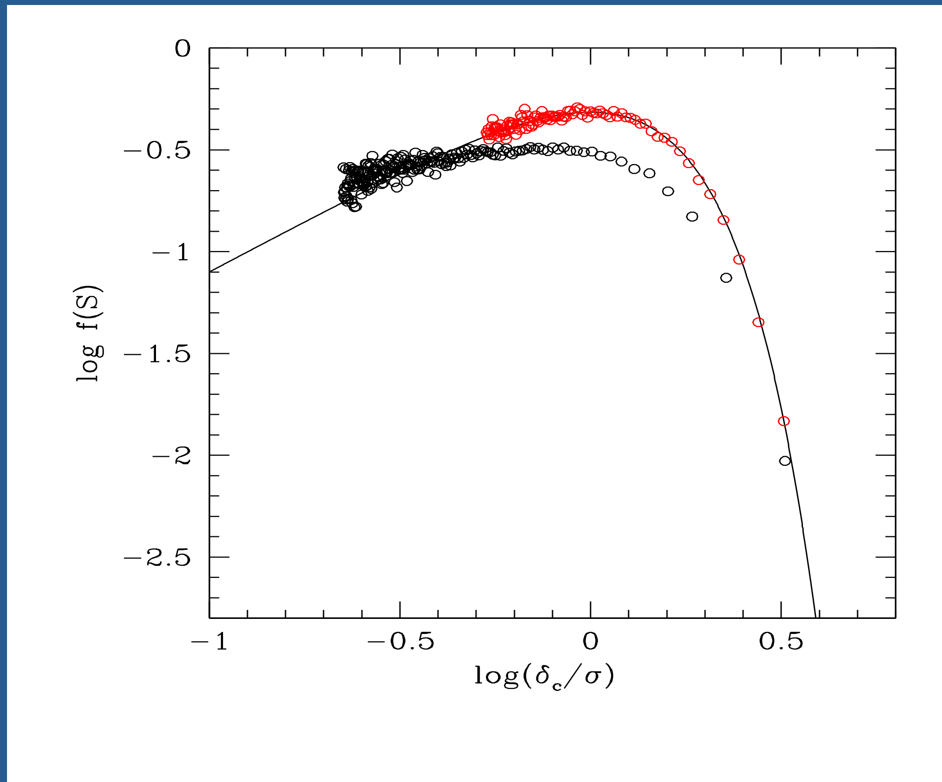
$$\frac{\partial \delta}{\partial \ln k} = Q(\ln k)\tilde{W}(k,R)$$

$$\langle Q(\ln k)\rangle = 0$$

$$\langle Q(\ln k)Q(\ln k')\rangle = \Delta^2(k)\delta_D(\ln k - \ln k')$$

Spherical Collapse Case:





Correlation Function

Generic Filter:

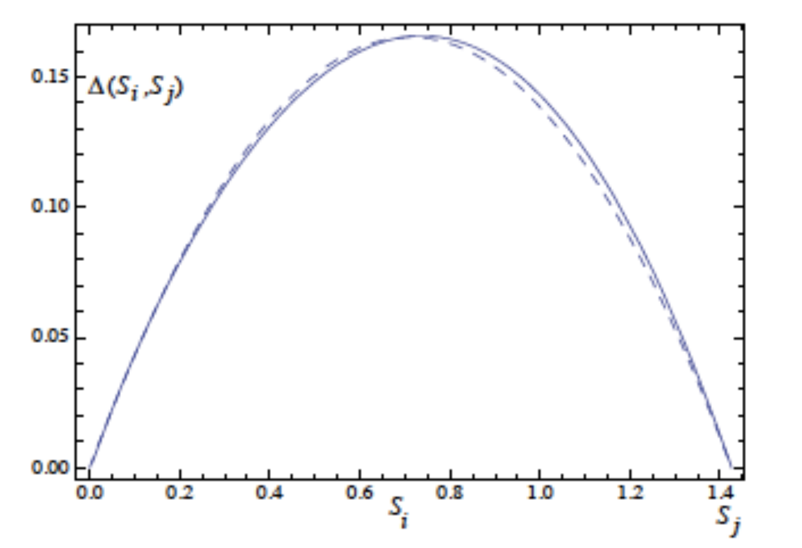
$$\langle \delta[R(S)]\delta[R(S')] \rangle = \frac{1}{2\pi^2}\int_0^\infty dk\ k^2 P(k) T^2(k) \tilde{W}[k,R(S)]\tilde{W}[k,R(S')]$$

Sharp-x Filter:

$$\langle \delta(S)\delta(S') \rangle = \int_0^S ds \int_0^{S'} ds' \langle \eta(s)\eta(s') \rangle = \min(S,S')$$

Introduce:

$$\Delta(S,S') = \langle \delta(S)\delta(S') \rangle - \min(S,S')$$



For LCDM power-spectrum:

$$\Delta(S,S') \cong \kappa \frac{S(S'-S)}{S'}$$ where κ≅ 0.47

Path-Integral Approach to Excursion Set

Discrete Random Walks

- Trajectory over a discrete "time" interval $\{Y_0, Y_1, \ldots, Y_n\}$ with $S_k = k\,\varepsilon$ and $k=1,..,n$

Ensemble Probability Density

$$p(Y_0,\ldots,Y_n,S_n) = \left\langle \delta_D[Y(S_1)-Y_1]\cdot\ldots\ldots\cdot\delta_D[Y(S_n)-Y_n]\right\rangle = \int D\lambda\; e^{i\sum \lambda_i Y_i} \left\langle e^{-i\sum \lambda_i Y(S_i)} \right\rangle$$

Partition function

$$e^{Z} = \left\langle e^{-i\sum \lambda_i Y(S_i)} \right\rangle$$

with

$$Z = \sum_{p=1}^{\infty} \frac{(-i)^p}{p!} \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} \lambda_{i_1} \ldots \lambda_{i_p} \left\langle Y(S_{i_1}) \ldots Y(S_{i_p}) \right\rangle_c$$

Connected Correlators

$$\left\langle Y(S_i) \right\rangle_c \equiv \overline{B}(S_i) = \delta_c + \beta\, S_i$$

$$\left\langle Y(S_i) Y(S_j) \right\rangle = (1 + D_B)\min(S_i, S_j) + \Delta(S_i, S_j)$$

κ-expansion around Markovian solution

Probability Distribution

$$\Pi_\varepsilon(Y_0,Y_n,S_n) = \int_0^\infty dY_1 \ldots \int_0^\infty dY_{n-1}\, p(Y_0,\ldots,Y_n,S_n)$$

Expansion to O(κ)

$$\Pi_\varepsilon(Y_0,Y_n,S_n) = \int_0^\infty dY_1 \ldots \int_0^\infty dY_{n-1} \int D\lambda \left(1 - \frac{1}{2}\sum_{i,j}\lambda_i\lambda_j\Delta_{ij}\right) e^{i\sum_k \lambda_k [Y_k - \bar{B}_k]}\, e^{-i\sum_{n,m}\lambda_n\lambda_m A_{nm}}$$

Isolate non-Markovian correction

$$\Pi_\varepsilon(Y_0,Y_n,S_n) = \Pi_\varepsilon^M(Y_0,Y_n,S_n) + \Pi_\varepsilon^{\kappa^{(1)}}(Y_0,Y_n,S_n)$$

Memory and Memory-of-Memory Terms

First Order Correction

$$\Pi_\varepsilon^{\kappa^{(1)}}(Y_0,Y_n,S_n) = \Pi_\varepsilon^{m}(Y_0,Y_n,S_n) + \Pi_\varepsilon^{m-m}(Y_0,Y_n,S_n)$$

$$\Pi_\varepsilon^{m}(Y_0,Y_n,S_n) = -\sum_{i=1}^{n-1}\Delta_{in}\partial_n\left[\Pi_\varepsilon^{M,f}(Y_0,0,S_i)\Pi_\varepsilon^{M,f}(0,Y_n,S_n-S_i)\right]$$

$$\Pi_\varepsilon^{m-m}(Y_0,Y_n,S_n) = \sum_{i<j}\Delta_{ij}\left[\Pi_\varepsilon^{M,f}(Y_0,0,S_i)\Pi_\varepsilon^{M,f}(0,0,S_j-S_i)\Pi_\varepsilon^{M,f}(0,Y_n,S_n-S_j)\right]$$

- Markovian solution around the barrier $\Pi_\varepsilon^{M,f}$

Continuous Limit

$$\sum_{i=1}^{n-1} \to \lim \frac{1}{\varepsilon}\int_0^S dS_i$$ & $$\sum_{i<j} \to \lim \frac{1}{\varepsilon^2}\int_0^S dS_i \int_{S_i}^S dS_j$$

Corrections to Mass Function

Memory

$$f_1^m(\sigma) = -2\sigma^2 \frac{\kappa Y_0}{(1+D_B)^2} \frac{\partial}{\partial S} \int_0^\infty dY_n \partial_n \left\{ Y_n e^{\frac{\beta}{1+D_B}(Y - Y_0 - \beta S/2)} Erfc\left[\frac{Y_0 + Y_n}{\sqrt{2S(1+D_B)}} \right] \right\} = 0$$

Memory-of-Memory

$$f_{1,\beta=0}^{m-m}(\sigma) = -\tilde{\kappa}\frac{\delta_c}{\sigma}\sqrt{\frac{2a}{\pi}}\left[e^{-\frac{a\delta_c^2}{2\sigma^2}} - \frac{1}{2}\Gamma\left(0, \frac{a\delta_c^2}{2\sigma^2}\right)\right]$$

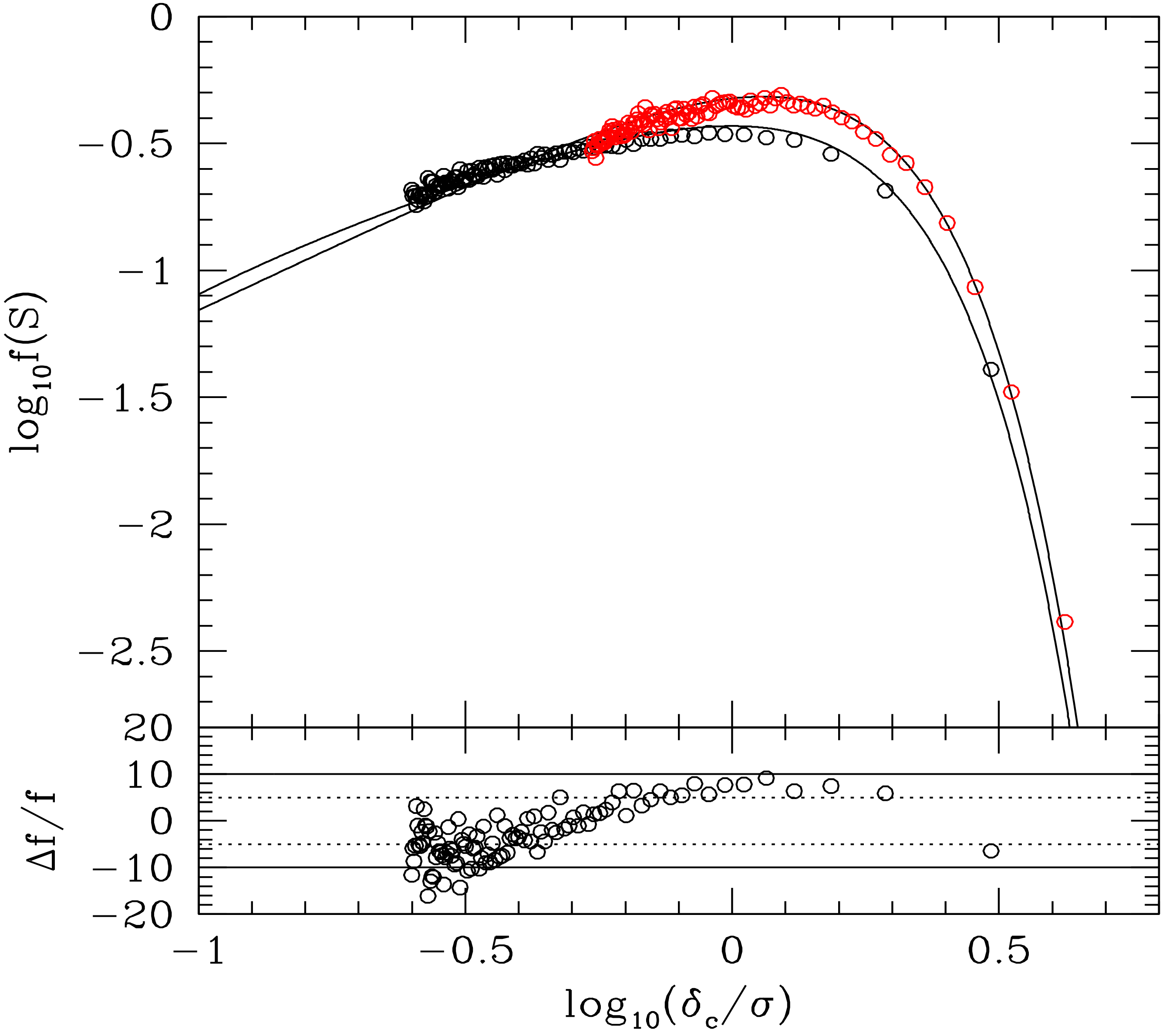
$$f_{1,\beta^{(1)}}^{m-m}(\sigma) = -\beta\, a\, \delta_c \left[f_{1,\beta=0}^{m-m}(\sigma) + \tilde{\kappa}\,\mathrm{Erfc}\left(\frac{\delta_c}{\sigma}\sqrt{\frac{a}{2}}\right)\right]$$

$$f_{1,\beta^{(2)}}^{m-m}(\sigma) = \beta^2\, a\, \delta_c\, \tilde{\kappa}\Big\{a\, \delta_c\, \mathrm{Erfc}\left(\frac{\delta_c}{\sigma}\sqrt{\frac{a}{2}}\right) +$$

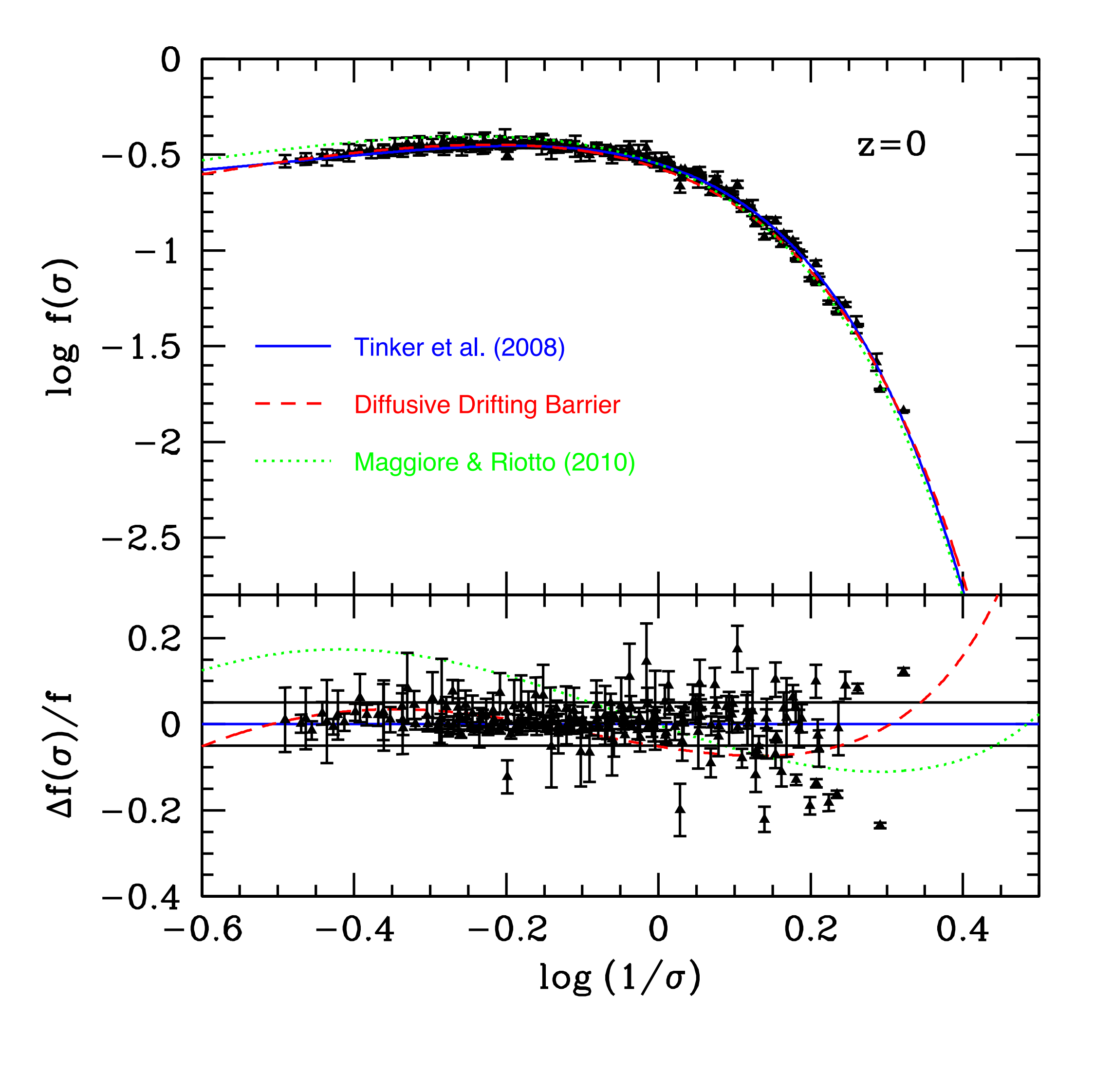
$$+\,\sigma\sqrt{\frac{a}{2\pi}}\left[e^{-\frac{a\delta_c^2}{2\sigma^2}}\left(\frac{1}{2} - \frac{a\delta_c^2}{\sigma^2}\right) + \frac{3}{4}\frac{a\delta_c^2}{\sigma^2}\Gamma\left(0, \frac{a\delta_c^2}{2\sigma^2}\right)\right]\Big\}$$

$$a = \frac{1}{1+D_B}$$

$$\tilde{\kappa} = \kappa a$$



0
−0.5
−1
−1.5
−2
−2.5
$\log_{10} f(S)$
20
10
0
−10
−20
$\Delta f/f$
−1
−0.5
0
0.5
$\log_{10}(\delta_c/\sigma)$



Calibration to Tinker et al.

- β=0.057
- D_B=0.294

If you are interested in non-CDM mass function: what to do?

1) **Solve linear perturbation theory and estimate S(R)**

   - Derive form of 2-pt correlator induced by the filter

2) **Write down non-linear collapse equation (spherical or ellipsoidal models) and estimate PDF(δ_{coll})**

   - Infer moments of the barrier statistics

3) **Solve Markovian System {δ,B}**

4) **Compute non-Markovian corrections using Path-Integral**

Conclusions

- DM Halo mass function is crucial in modern cosmology, physical understanding is needed
- Excursion Set is a self-consistent framework for a theoretical modeling of MF
- Path-Integral Methods allows analytical computation of generic filter corrections and consistent comparison with N-body results
- Focus on implementing physical models of halo collapse and properties of invisible components