Excursion Set Theory

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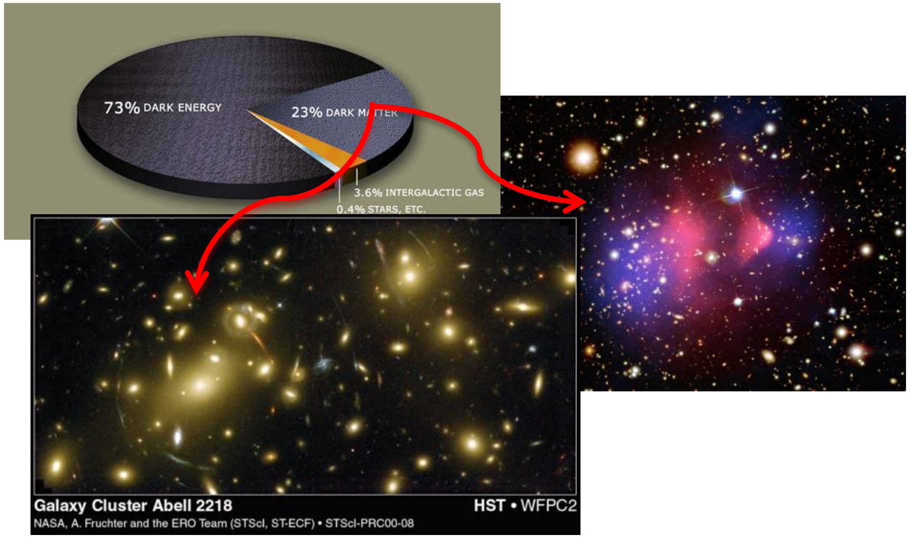
Cosmic Structures and Dark Matter

Dark Matter:

- Foster matter clustering
- Resides in virialized clumps



The Role of Halos:

- Building blocks of cosmic structure formation
- Shape baryon distribution

Halo Properties & Mass Function:

- Statistics of the initial density field
- Non-linear gravitational collapse

- Nature DM particles
- Underlying cosmology

Numerical N-body Simulations Results:

- Collisionless Particle Algorithms (CDM)
- Still require theoretical understanding

$$f(\sigma) = A\left[\left(\frac{\sigma}{b}\right)^{-a} + 1\right] e^{-c/\sigma^2},$$

Outline

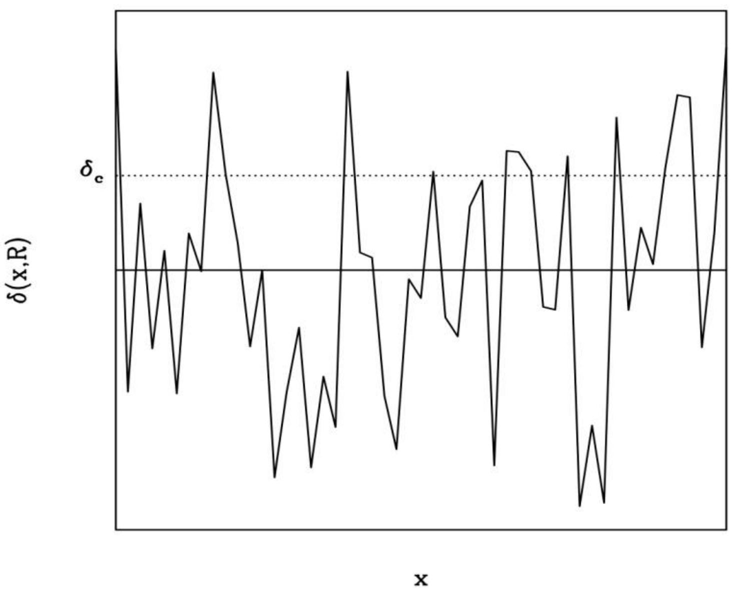
- Press-Schechter Formalism
- The Excursion Set Theory
- Halo Collapse Model
- Halo Mass Definition & Non-Markovianity
- Path-Integral Approach
- Comparison with N-body results

Reference List

- Bond et al., ApJ 379, 440 (1991)
- Sheth, Mo & Tormen, MNRAS 323, 1 (2001)
- Percival, MNRAS 327, 1313 (2001)
- Maggiore & Riotto, ApJ 711, 907 (2010)
- Maggiore & Riotto, ApJ 717, 515 (2010)
- Maggiore & Riotto, ApJ 717, 526 (2010)
- De Simone, Maggiore & Riotto, MNRAS 412, 2587 (2011)
- Ma et al., MNRAS 411, 2644 (2011)
- Corasaniti & Achitouv, PRL 106, 241302 (2011)
- Corasaniti & Achitouv, PRD 84, 023009 (2011)

Press-Schechter Formalism

Halo Mass Distribution & Linear Density Field:



- Filtering scale and mass

$$M = \bar{\rho}\; V(R)$$

- Statistics of the smoothed density field

$$\Pi(\delta, R)$$

- Linearly extrapolated collapse threshold

$$\delta_c$$

- Fraction of mass in halos > M

$$F_{PS}(R[M]) = \int_{\delta_c}^{\infty} d\delta\, \Pi(\delta, R[M])$$

- #-halos with mass dM

$$\frac{dn}{dM} = \frac{1}{V}\frac{dF}{dM}$$

Press-Schechter Mass Function

Gaussian Field:

- Variance of the smoothed field

$$\sigma^2(R) \equiv S(R) = \frac{1}{2\pi^2}\int k^2 P(k)\left|\tilde{W}(k,R)\right|^2 dk$$

- PDF

$$\Pi(\delta, S[R]) = \frac{1}{\sqrt{2\pi S[R]}}\, e^{-\delta^2/2S[R]}$$

- Spherical collapse threshold

$$\delta_c$$

- Fraction of mass in halos

$$F_{PS}(R) = \int_{\delta_c}^{\infty} d\delta\, \Pi(\delta, S(R)) = \frac{1}{2} Erfc\left[\frac{\delta_c}{\sigma(R)\sqrt{2}}\right]$$

- Halo Mass Function

$$\frac{dn}{dM} = \frac{\bar{\rho}}{M^2}\frac{d\ln\sigma^{-1}}{d\ln M} f(\sigma)$$

with

$$f_{PS}(\sigma) = 2\sigma^2 \frac{dF_{PS}}{dS} = \frac{1}{\sqrt{2\pi}}\frac{\delta_c}{\sigma} e^{-\delta_c^2/(2\sigma^2)}$$

Cloud-in-Cloud Problem

Asymptotic Behavior:

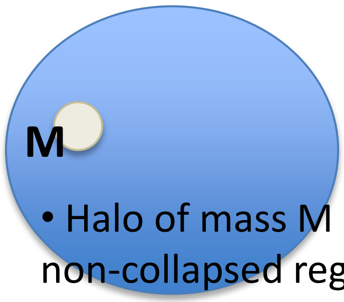
- In the limit R → 0 all mass must be in collapsed structures, F(0)=1

- In the PS calculation half of the mass is miscounted

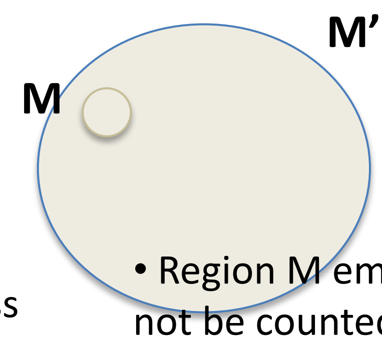
$$F_{PS}(R) = \frac{1}{2} Erfc\left[\frac{d_c}{s(R)\sqrt{2}}\right] \xrightarrow[R\to 0]{} \frac{1}{2}$$

Problem: in the PS approach there is no mass ordering

- No distinction between different configurations (cloud-in-cloud)



- Halo of mass M embedded in a larger non-collapsed region contributes to mass function at mass M



- Region M embedded in halo M'>M. M should not be counted in mass function since already included at M'

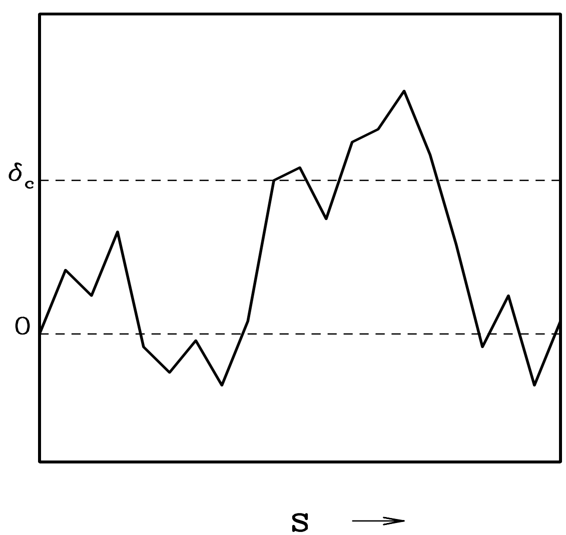
Excursion Set Theory

$$\delta(x,R) = \frac{1}{(2\pi)^3}\int d^3k\ \delta(k)\tilde{W}(k,R)\ e^{-ikx}$$

- At any point x, δ performs a random walk as function of R

- Langevin Equation:

$$\frac{\partial \delta}{\partial R} = V(R) \ \text{ and } \ V(R) = \frac{1}{(2\pi)^3}\int d^3k\ \delta(k)\frac{\partial \tilde{W}}{\partial R}e^{-ikx}$$



- random walks start at R = ∞ (S=0) with δ = 0 evolving toward smaller R (larger S)

- ς(R) depends on Π(δ) and W(x,R)

- Halos of mass M corresponds to trajectories crossing the threshold at S(M)

- cloud-in-cloud solved by requiring first crossing

Excursion Set Mass Function

Stochastic Problem:

- Computation of the probability distribution of random walks with absorbing boundary, $\Pi(\delta,\delta_c,S)$
- Multiplicity function obtained from the first-crossing rate $\mathcal{F}(S)=dF/dS$

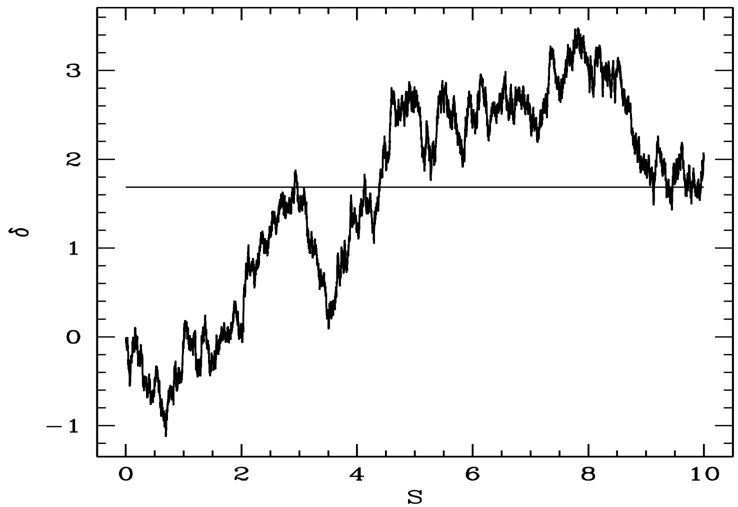
$$\int_0^S \mathcal{F}(S')\,dS' = 1 - \int_{-\infty}^{\delta_c} \Pi(\delta,\delta_c,S)\,d\delta \quad \Longrightarrow \quad \frac{dF}{dS} = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c} \Pi(\delta,\delta_c,S)\,d\delta$$

Cloud-in-cloud: $\lim_{S\to\infty} \Pi(\delta,\delta_c,S) = 0$

Sharp-k filter: $\tilde{W}(k,R) = \theta(1/R - k)$

- Markovian random walks

$$\frac{\partial \delta}{\partial S} = \eta(S) \quad \text{with} \quad \langle \eta(S)\rangle = 0, \quad \langle \eta(S)\eta(S')\rangle = \delta_D(S-S')$$



Extended Press-Schechter

Fokker-Planck Equation:

$$\frac{\partial \Pi}{\partial S} = \frac{1}{2}\frac{\partial^2 \Pi}{\partial \delta^2} \qquad \text{with} \qquad \begin{aligned} \Pi(\delta, 0) &= \delta_D(\delta) \\ \Pi(\delta_c, S) &= 0 \end{aligned}$$

Solution:

$$\Pi(\delta, S) = \frac{1}{\sqrt{2\pi S}}\left[e^{-\delta^2/(2S)} - e^{-(2\delta_c-\delta)^2/(2S)}\right] \qquad \text{for } \delta < \delta_c$$

Multiplicity Function:

$$\frac{dF}{dS} = -\frac{\partial}{\partial S}\int_{-\infty}^{\delta_c} \Pi(\delta, \delta_c, S)\, d\delta \quad \Longrightarrow \quad f_{EPS}(\sigma) = \sqrt{\frac{2}{\pi}}\,\frac{\delta_c}{\sigma}\, e^{-\delta_c^2/(2\sigma^2)}$$

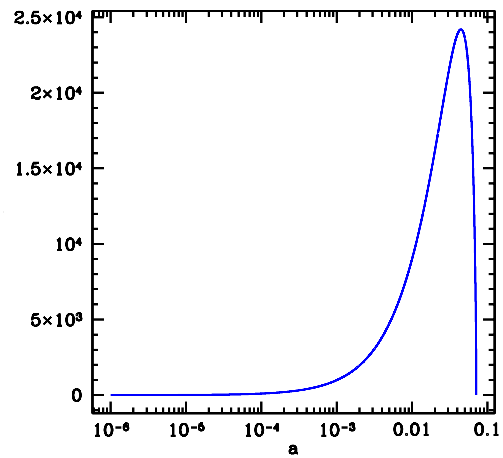
Halo Collapse Model

Spherical Collapse (Gunn & Gott, 1973)

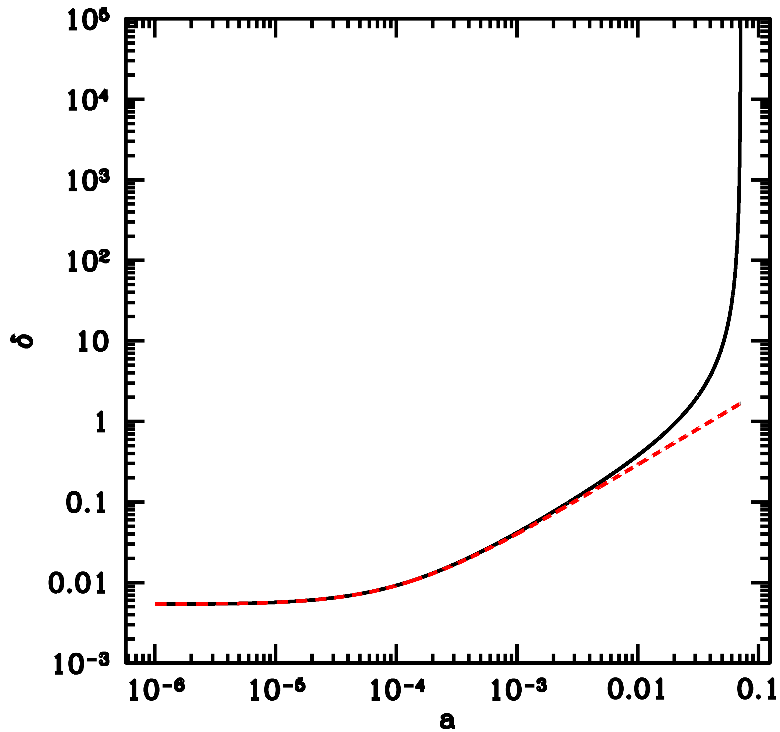
- Top-hat perturbation in FRW background, y=R/R_i

$$\ddot{y} = -\frac{4}{3}\pi G\bar{\rho}^i_m(1+\delta^i_m)\frac{1}{y^2}$$

- Dynamics independent of R_i



$$1+\delta_m = (1+\delta^i_m)y^3\left(\frac{a}{a_i}\right)^3$$



Non-Spherical Halo Collapse

Ellipsoidal Collapse

- Initial Gaussian fluctuations are non-spherical (Doroshkevich, 1970)
- Ellipsoidal halos collapse and shear

$$\frac{d^2 a_i}{dt^2} = \frac{8}{3}\pi G\bar{\rho}_\Lambda a_i - 4\pi G\bar{\rho}_m a_i \left[\frac{1}{3} + \frac{\Delta(t)}{3} + \frac{b_i'(t)}{2}\Delta(t) + \lambda_i'(t)\right]$$

$$b_i' = -\frac{2}{3} + a_1 a_2 a_3 \int_0^\infty \frac{d\tau}{(a_i^2+\tau)\Pi_{m=1}^3 (a_m^2+\tau)^{1/2}}$$

$$\lambda_1 = \frac{\delta}{3}(1-3e+p)$$

$$\lambda_2 = \frac{\delta}{3}(1-2p)$$

$$\lambda_3 = \frac{\delta}{3}(1+3e+p)$$

- Dynamics dependent of initial size of the collapsing region
- Critical Overdensity is mass dependent (e.g. Eisenstein & Loeb, 1995)

"Fuzzy" Barrier

Stochastic Barrier

- Ellipsoidal parameters are random variables with characteristic probability distribution

- e.g. for a Gaussian density field
$$g(e,p|\delta) = \frac{1125}{\sqrt{10\pi}} e(e^2-p^2)\left(\frac{\delta}{\sigma}\right)^5 \exp\left[-\frac{5}{2}\frac{\delta^2}{\sigma^2}(3e^2+p^2)\right]$$

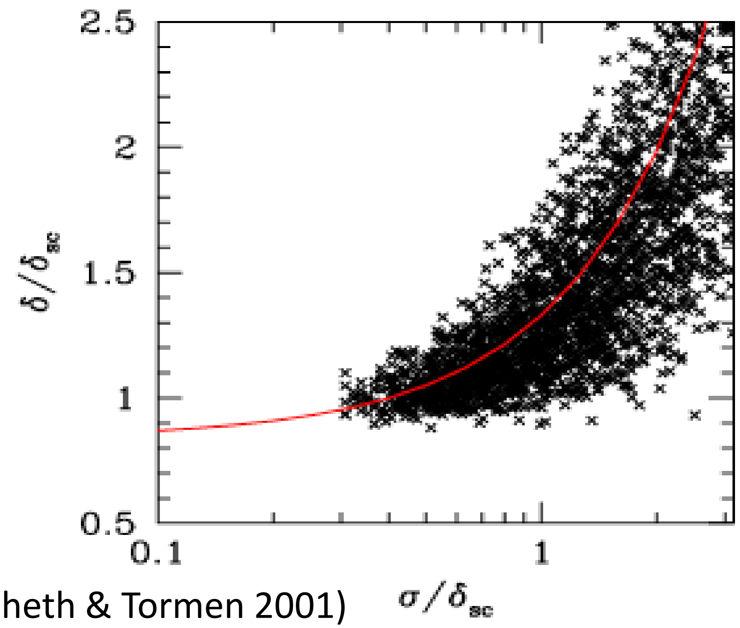
- Density threshold is a random variable

- e.g $\langle B(S)\rangle = \delta_c[1+\beta\,(S/S_*)^\gamma]$

(Sheth, Mo & Tormen 2001)

- Stochastic barrier model: specify the moments of barrier's PDF

$\langle B(S)\rangle$ & $\langle (B(S)-\langle B(S)\rangle)^2\rangle$, ...



(Sheth & Tormen 2001)

Stochastic Barrier and Excursion Set

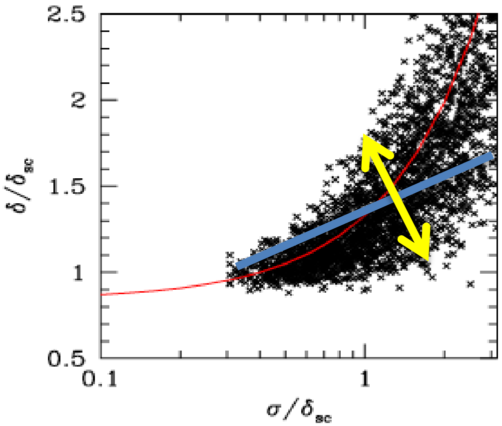
Non-Spherical Collapse:

- Diffusive Drifting Barrier $\langle B(S)\rangle = \delta_c + \beta S$ $\quad \langle [B(S) - \langle B(S)\rangle]^2 \rangle^{1/2} = \sqrt{D_B}\, s$

β = rate of average deviation from spherical collapse

D_B = scatter of the collapse condition around mean (Maggiore & Riotto 2010b)

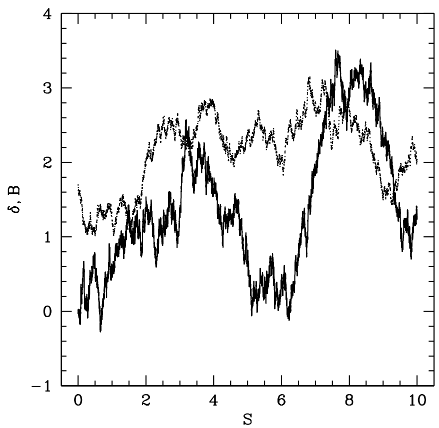
- Introduce: Y=B-δ



Sharp-k filter case:

$$\frac{\partial Y}{\partial S} = \beta + h(S) \qquad \langle h(S)\rangle = 0$$

$$\langle h(S)h(S')\rangle = (1+D_B)\,\delta_D(S-S')$$



Langevin & Fokker-Planck Equations

Generic Gaussian Random Walk:

$$\frac{\partial Y}{\partial S} = h(S) \qquad \text{with} \qquad \begin{aligned} \langle h(S) \rangle &= A(Y,S) \\ \langle h(S)h(S') \rangle &= B(Y,S) \end{aligned}$$

Probability Distribution Obeys:

$$\frac{\partial \mathrm{P}}{\partial S} = -\frac{\partial}{\partial Y}\left[A(Y,S)\mathrm{P}(Y,S)\right] + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\left[B(Y,S)\mathrm{P}(Y,S)\right]$$

$$\frac{\partial \Pi}{\partial S} = -\beta \frac{\partial \Pi}{\partial Y} + \frac{1+D_B}{2}\frac{\partial^2 \Pi}{\partial Y^2}$$

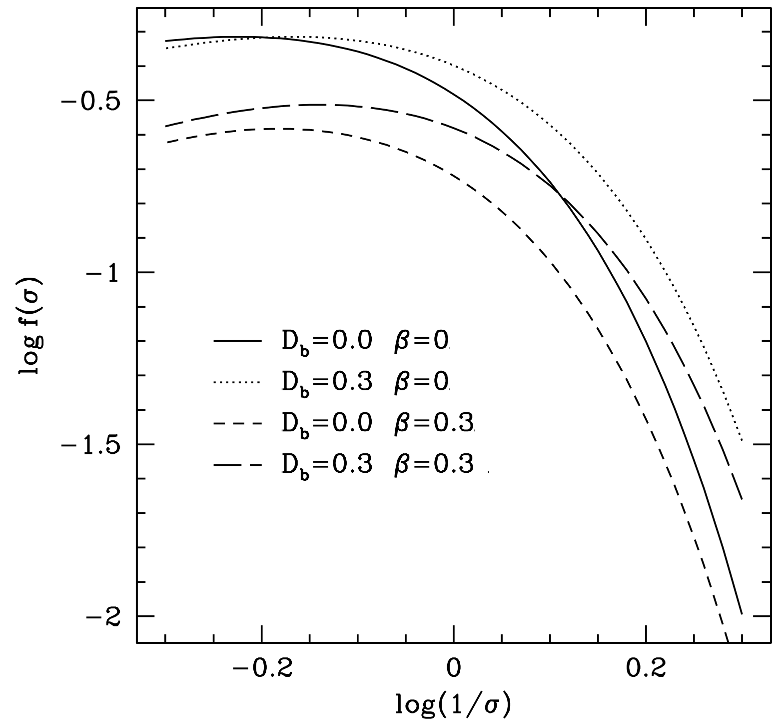
$$\Pi(Y,0) = \delta_D(Y - \delta_c)$$

$$\Pi(0,S) = 0$$

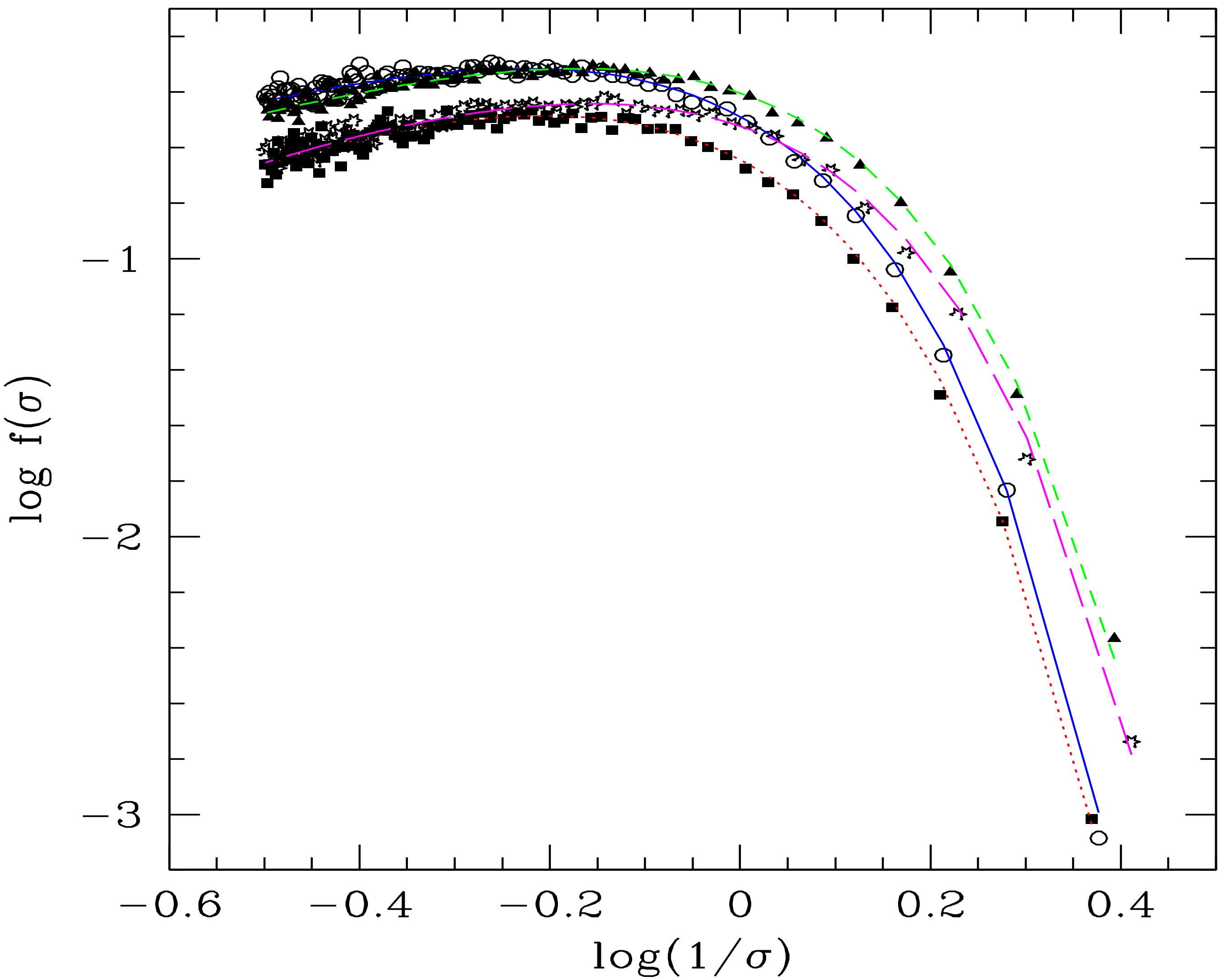
$$\Pi(Y,S) = \frac{e^{\frac{\beta}{1+D_B}(Y - Y_0 - \beta S/2)}}{\sqrt{2\pi S(1+D_B)}}\left[e^{-\frac{(Y-Y_0)^2}{2S(1+D_B)}} - e^{-\frac{(Y+Y_0)^2}{2S(1+D_B)}} \right]$$

Multiplicity Function:

$$f(\sigma) = \sqrt{\frac{2}{\pi}}\frac{\delta_c}{\sigma\sqrt{1+D_B}} e^{-\frac{(\delta_c+\beta\sigma^2)^2}{2\sigma^2(1+D_B)}}$$



(PSC & I. Achitouv 2011a,b)



log f(σ)
−1
−2
−3
−0.6
−0.4
−0.2
0
0.2
0.4
log(1/σ)

Filter Function and Halo Mass Definition

Mass and smoothing scale: $M(R) = \bar{\rho}\, V(R)$ with $V(R) = \int d^3x\, W(x,R)$

- Unambiguously define only for sharp-x filter:

$$V(R) = \int d^3x\, W(x,R) = 4/3\, \pi R^3$$

- Generic filters define M up to an integration constant
- Sharp-k leaves M undefined

$$V(R) = 12\, \pi R^3 \int_0^\infty du \left[\frac{\sin u}{u} - \cos u \right]$$

- Sharp-x leads to correlated random walks since

$$\tilde{W}(k,R) = 3\, \frac{\sin(kR) - (kR)\cos(kR)}{(kR)^3}$$

Inaccuracies EPS result

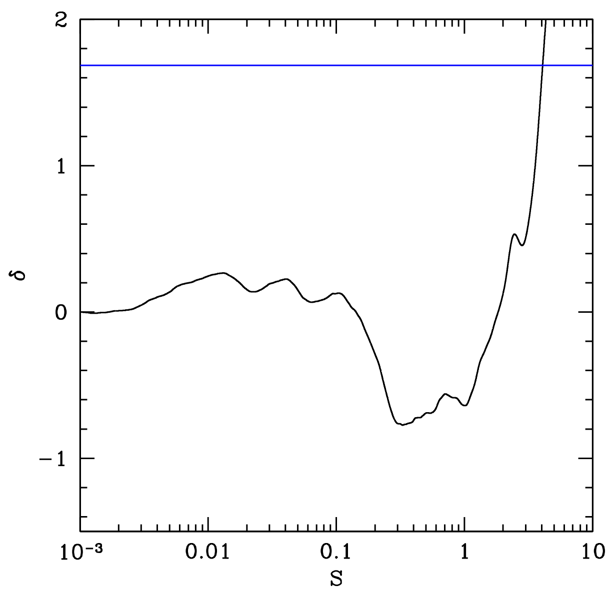
Langevin Equation:

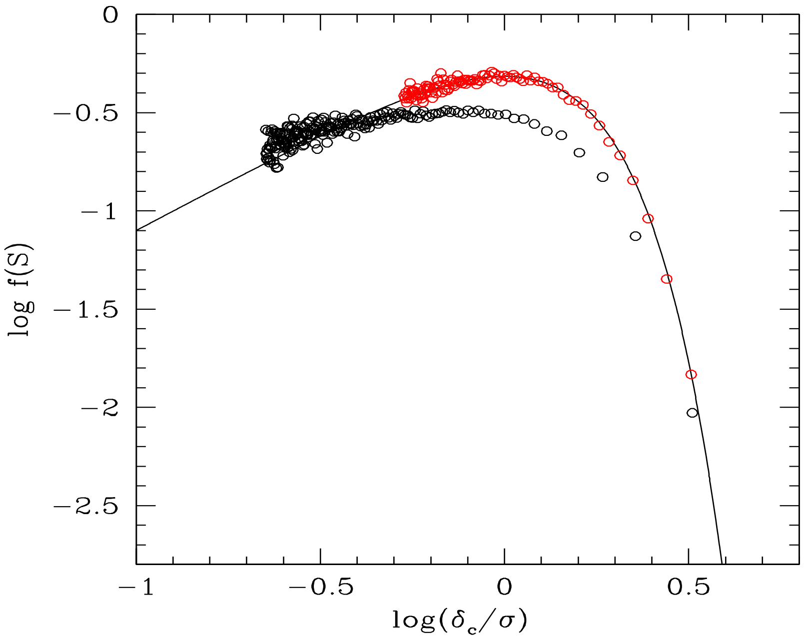
$$\frac{\partial \delta}{\partial \ln k} = Q(\ln k)\tilde{W}(k, R)$$

$$\langle Q(\ln k) \rangle = 0$$

$$\langle Q(\ln k) Q(\ln k') \rangle = \Delta^2(k)\,\delta_D(\ln k - \ln k')$$

Spherical Collapse Case:





Correlation Function

(Maggiore & Riotto 2010a)

Generic Filter:

$$\langle \delta[R(S)]\delta[R(S')]\rangle = \frac{1}{2\pi^2}\int_0^\infty dk\, k^2 P(k) T^2(k) \tilde{W}[k,R(S)]\tilde{W}[k,R(S')]$$

Sharp-k Filter:

$$\langle \delta(S)\delta(S')\rangle = \int_0^S ds \int_0^{S'} ds' \langle \eta(s)\eta(s')\rangle = \min(S,S')$$

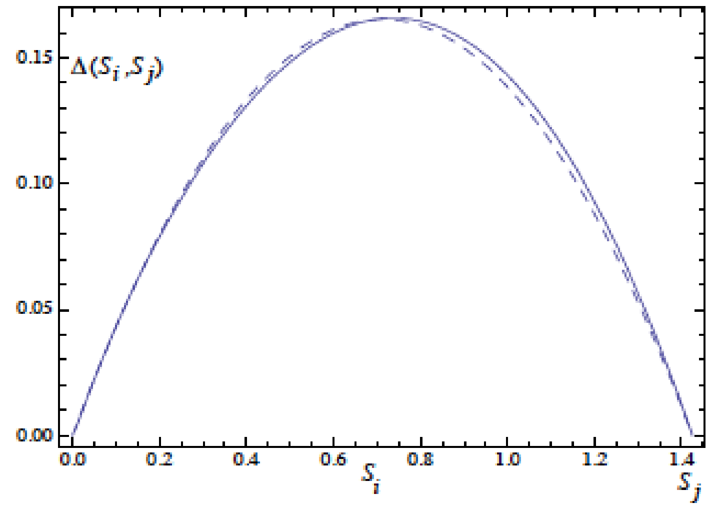
Introduce:

$$\Delta(S,S') = \langle \delta(S)\delta(S')\rangle - \min(S,S')$$

For LCDM power-spectrum:

$$\Delta(S,S') \approx \kappa \frac{S(S'-S)}{S'}$$

where κ≅ 0.47



Path-Integral Approach to Excursion Set

(Maggiore & Riotto 2010a)

Discrete Random Walks

- Trajectory over a discrete "time" interval $\{Y_0, Y_1, \ldots, Y_n\}$ with $S_k = k\,\varepsilon$ and $k=1,..,n$

Ensemble Probability Density

$$p(Y_0,\ldots,Y_n,S_n) = \left\langle \delta_D[Y(S_1)-Y_1]\cdot\ldots\cdot\delta_D[Y(S_n)-Y_n]\right\rangle = \int D\lambda\; e^{i\sum \lambda_i Y_i}\left\langle e^{-i\sum \lambda_i Y(S_i)}\right\rangle$$

Partition function

$$e^{Z} = \left\langle e^{-i\sum \lambda_i Y(S_i)}\right\rangle \quad \text{with} \quad Z = \sum_{p=1}^{\infty}\frac{(-i)^p}{p!}\sum_{i_1=1}^{n}\ldots\sum_{i_p=1}^{n}\lambda_{i_1}\ldots\lambda_{i_p}\left\langle Y(S_{i_1})\ldots Y(S_{i_p})\right\rangle_c$$

Connected Correlators

$$\left\langle Y(S_i)\right\rangle_c \equiv \bar{B}(S_i) = \delta_c + \beta\, S_i$$

$$\left\langle Y(S_i)Y(S_j)\right\rangle = (1+D_B)\min(S_i,S_j) + \Delta(S_i,S_j)$$

κ-expansion around Markovian solution

(Maggiore & Riotto 2010a)

Probability Distribution

$$\frac{dF}{dS} = -\frac{\partial}{\partial S}\int_0^{\infty} \Pi_{RW}(Y, Y_0, S)\, d\delta$$

$$\Pi_\varepsilon(Y_0, Y_n, S_\eta) = \int_0^{\infty} dY_1 \ldots \int_0^{\infty} dY_{n-1}\, p(Y_0, \ldots, Y_n, S_\eta)$$

Expansion to O(κ)

$$\Pi_\varepsilon(Y_0, Y_n, S_\eta) = \int_0^{\infty} dY_1 \ldots \int_0^{\infty} dY_{n-1} \int D\lambda \left(1 - \frac{1}{2}\sum_{i,j}\lambda_i \lambda_j \Delta_{ij}\right) e^{i\sum_k \lambda_k [Y_k - \bar{B}_k]}\, e^{-i\sum_{n,m}\lambda_n \lambda_m A_{nm}}$$

Markovian solution and non-Markovian corrections

$$\Pi_\varepsilon(Y_0, Y_n, S_\eta) = \Pi_\varepsilon^{M}(Y_0, Y_n, S_\eta) + \Pi_\varepsilon^{\kappa^{(1)}}(Y_0, Y_n, S_\eta)$$

Markovian solution & Chapman-Kolmogorov

$$\Pi_{\varepsilon}^{M}(Y_0, Y_n, S_n) = \int_0^{\infty} dY_1 \ldots \int_0^{\infty} dY_{n-1}\, p_0(Y_0,\ldots, Y_n, S_n)$$

Markovian Density: $$p_0(Y_0,\ldots, Y_n, S_n) = \int D\lambda\; e^{\,i\sum_k \lambda_k [Y_k - \bar{B}_k]}\; e^{\,-i\sum_{n,m} \lambda_n \lambda_m A_{nm}}$$

with $A_{ij} = \min(i, j)$

$$p_0(Y_0,\ldots, Y_n, S_n) = \frac{1}{\left[2\pi\varepsilon(1+\Delta_B)\right]^{\frac{n}{2}}}\, e^{-\frac{1}{2\varepsilon(1+\Delta_B)}\sum_{i=1}^{n-1}\left[(Y_{i+1}-Y_i)-(\bar{B}_{i+1}-\bar{B}_i)\right]^2}$$

$$= \psi_{\varepsilon}(\Delta Y)\, p_0(Y_0,\ldots, Y_{n-1}, S_{n-1})$$

with $$\psi_{\varepsilon}(\Delta Y) = \frac{1}{\sqrt{2\pi\varepsilon(1+\Delta_B)}}\, e^{-\frac{(\Delta Y - \beta\varepsilon)^2}{2\varepsilon(1+\Delta_B)}}$$

Chapman-Kolmogorov: $$\Pi_{\varepsilon}^{M}(Y_0, Y_n, S_n) = \int_0^{\infty} dY_{n-1}\, \psi_{\varepsilon}(\Delta Y)\, \Pi_{\varepsilon}^{M}(Y_0, Y_{n-1}, S_{n-1})$$

- In the continous limit and developing RHS in S+ε and LHS in Y-ΔY we recover:

$$\frac{\partial \Pi^{M}_{\varepsilon=0}}{\partial S} = -\beta\frac{\partial \Pi^{M}_{\varepsilon=0}}{\partial Y} + \frac{1+\Delta_B}{2}\frac{\partial^2 \Pi^{M}_{\varepsilon=0}}{\partial Y^2}$$

Memory and Memory-of-Memory Terms

First Order Corrections:

$$P_e^{k(1)}(Y_0, Y_n, S_n) = P_e^{m}(Y_0, Y_n, S_n) + P_e^{m-m}(Y_0, Y_n, S_n)$$

Memory:

$$P_e^{m}(Y_0, Y_n, S_n) = -\sum_{i=1}^{n-1} D_{in}\partial_n\left[P_e^{M,f}(Y_0, 0, S_i)P_e^{M,f}(0, Y_n, S_n - S_i)\right]$$

Memory-of-Memory:

$$P_e^{m-m}(Y_0, Y_n, S_n) = \sum_{i<j} D_{ij}\left[P_e^{M,f}(Y_0, 0, S_i)P_e^{M,f}(0, 0, S_j - S_i)P_e^{M,f}(0, Y_n, S_n - S_j)\right]$$

- Markovian solution around the barrier $P_e^{M,f}$

Continuous Limit:

$$\sum_{i=1}^{n-1} \rightarrow \lim \frac{1}{e}\int_0^S dS_i \quad \& \quad \sum_{i<j} \rightarrow \lim \frac{1}{e^2}\int_0^S dS_i \int_{S_i}^S dS_j$$

Corrections to Mass Function

Memory

$$f_1^m(\sigma) = -2\sigma^2 \frac{\kappa Y_0}{(1+\Delta_B)^2} \frac{\partial}{\partial S} \int_0^{\infty} dY_n \partial_n \left\{ Y_n e^{\frac{\beta}{1+\Delta_B}(Y - Y_0 - \beta S/2)} Erfc\left[\frac{Y_0 + Y_n}{\sqrt{2S(1+\Delta_B)}} \right] \right\} = 0$$

Memory-of-Memory

$$f^{m-m}_{1,\beta=0}(\sigma) = -\tilde{\kappa}\frac{\delta_c}{\sigma}\sqrt{\frac{2a}{\pi}}\left[e^{-\frac{a\delta_c^2}{2\sigma^2}} - \frac{1}{2}\Gamma\left(0, \frac{a\delta_c^2}{2\sigma^2}\right)\right]$$

$$f^{m-m}_{1,\beta^{(1)}}(\sigma) = -\beta\, a\, \delta_c \left[f^{m-m}_{1,\beta=0}(\sigma) + \tilde{\kappa}\, \mathrm{Erfc}\left(\frac{\delta_c}{\sigma}\sqrt{\frac{a}{2}}\right)\right]$$

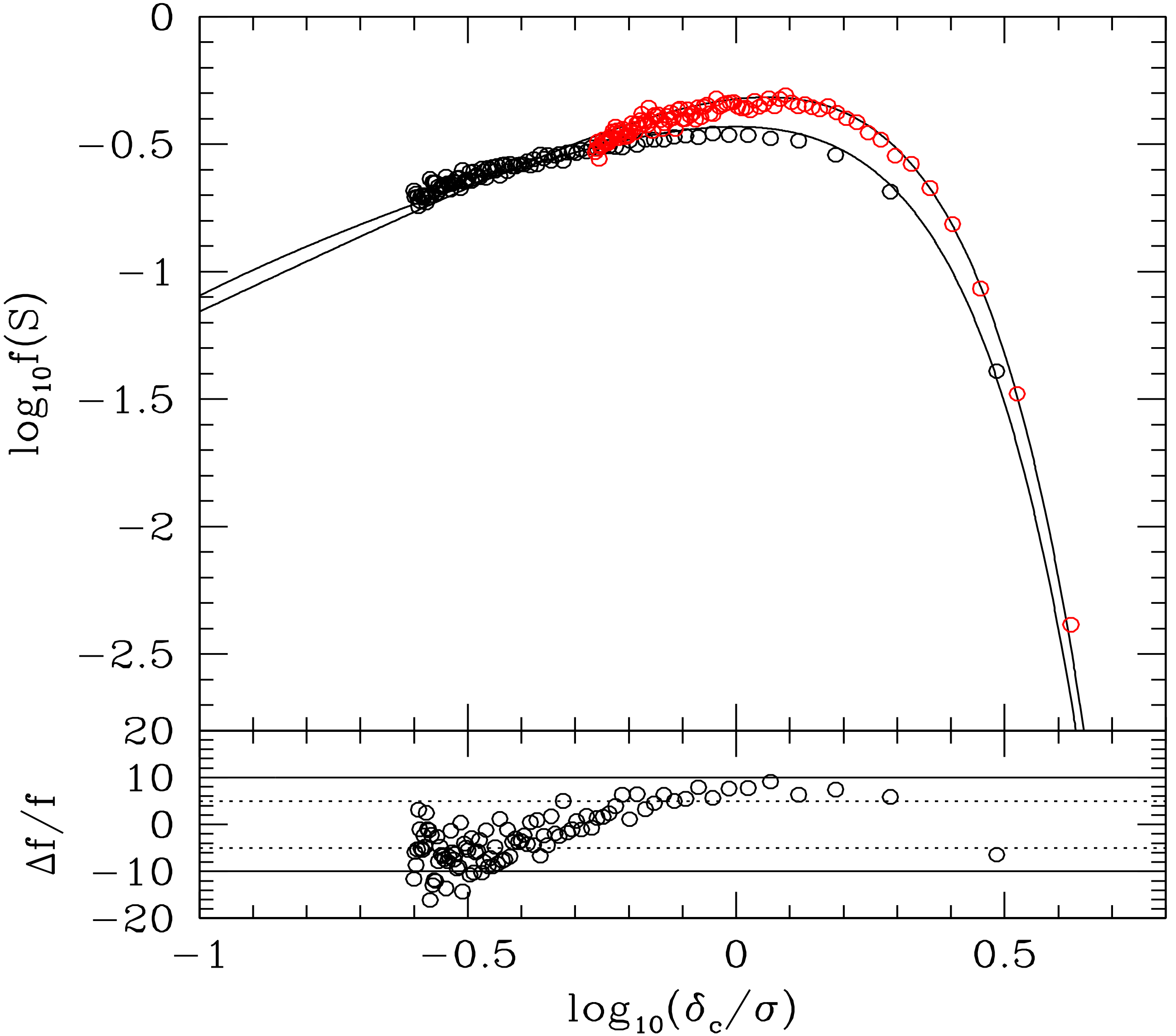
$$a = \frac{1}{1+\Delta_B}$$

$$f^{m-m}_{1,\beta^{(2)}}(\sigma) = \beta^2\, a\, \delta_c\, \tilde{\kappa}\left\{a\, \delta_c\, \mathrm{Erfc}\left(\frac{\delta_c}{\sigma}\sqrt{\frac{a}{2}}\right) + \right.$$

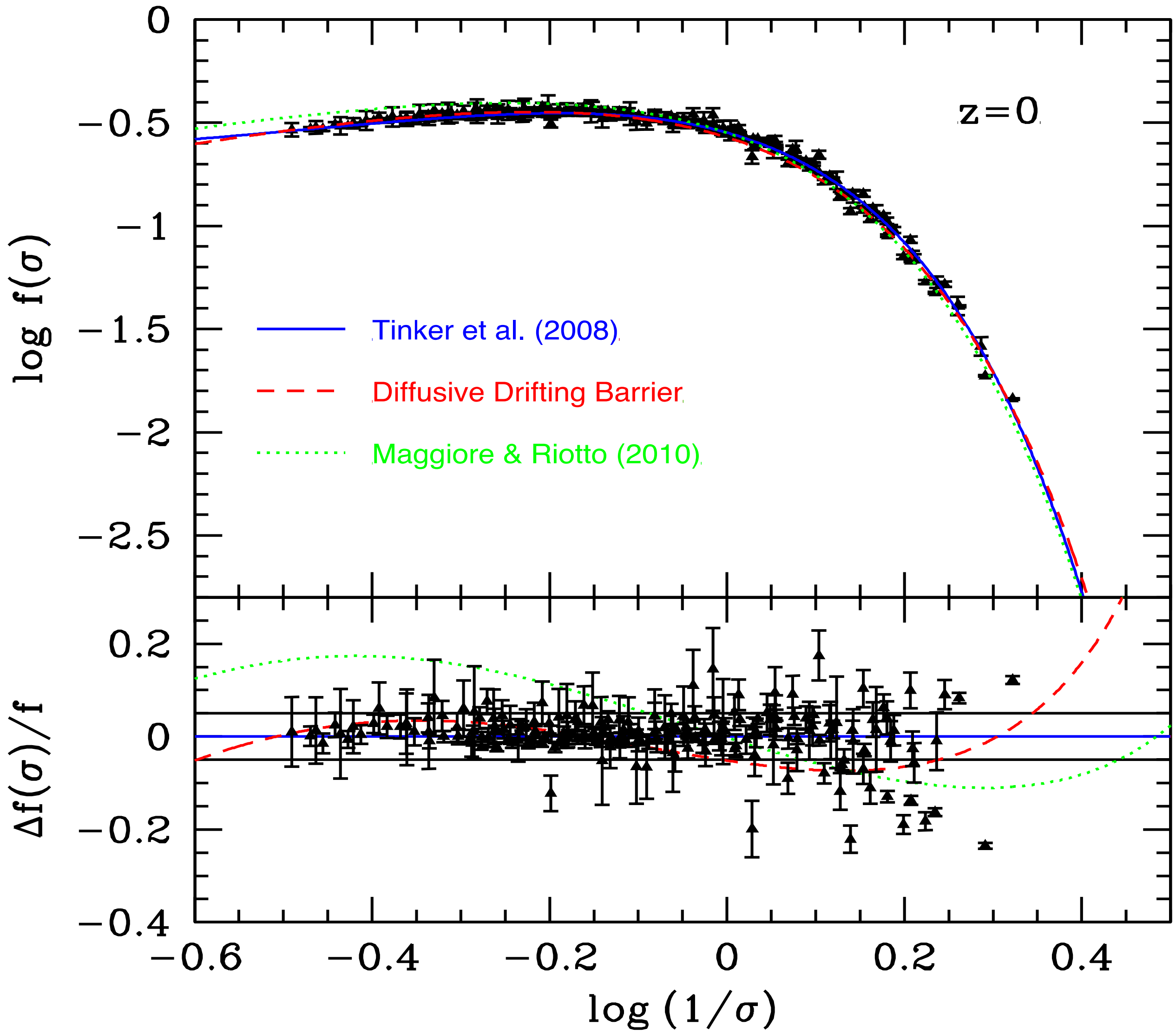
$$\left. + \sigma\sqrt{\frac{a}{2\pi}}\left[e^{-\frac{a\delta_c^2}{2\sigma^2}}\left(\frac{1}{2} - \frac{a\delta_c^2}{\sigma^2}\right) + \frac{3}{4}\frac{a\delta_c^2}{\sigma^2}\Gamma\left(0, \frac{a\delta_c^2}{2\sigma^2}\right)\right]\right\}$$

$$\tilde{\kappa} = \kappa a$$

(PSC & I. Achitouv 2011a,b)



0
−0.5
−1
−1.5
−2
−2.5
20
10
0
−10
−20
$\log_{10} f(S)$
$\Delta f / f$
−1
−0.5
0
0.5
$\log_{10}(\delta_c/\sigma)$



(PSC & I. Achitouv 2011a,b)

Conclusions

- Path-Integral formulation of Excursion Set theory and Halo Collapse model

- Predicts barrier cosmology dependence from Ellipsoidal Collapse Model

- Extension to models beyond CDM

- Halo forms mostly on initial density peaks and not on random points: how accurate can the barrier model be?