Analytic approaches for large-scale structures

Warm Dark Matter late-time velocity dispersion

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Impact of a Warm Dark Matter late-time velocity dispersion on the cosmic web

P.V., arXiv:1206.0554

A- Specific features of WDM

- Collisionless dark matter particles as for CDM

- "Low mass" (1 keV) ⇒ significant velocity dispersion and free-streaming

⇒ small-scale density fluctuations are erased by free-streaming (mostly during the relativistic era)

⇒ high-k cutoff for the (linear) density power spectrum

- This may solve some small-scale problems of CDM (galaxy satellites, cores/cusps, ..) (but baryon processes may be an alternative)

B- Dynamics

I) Analytic description at early times

"Warm": finite velocity dispersion → phase-space distribution function $f(\mathbf{x},\mathbf{p},\tau)$

In the non-relativistic regime, the system is described by the Vlasov-Poisson system:

$$\frac{\partial f}{\partial \tau} + \frac{\mathbf{p}}{ma} \cdot \frac{\partial f}{\partial \mathbf{x}} - ma \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$$

$$\Delta \Phi = \frac{4\pi \mathcal{G} m}{a} \left(\int d\mathbf{p}\, f - \bar{n} \right)$$

Self-gravity → Non-linear → Linearize the Vlasov equation

In practice, one is mostly interested in the density and peculiar velocity fields.
Can one simplify the description ?

density: $\int d\mathbf{p}\, f = \frac{\rho}{m}$

(mean) velocity: $\int d\mathbf{p}\, p_i\, f = a\rho\, v_i$

velocity dispersion: $\int d\mathbf{p}\, p_i p_j\, f = m a^2 \rho\, (v_i v_j + \sigma_{ij})$

Velocity moments of $\frac{\partial f}{\partial \tau} + \frac{\mathbf{p}}{ma} \cdot \frac{\partial f}{\partial \mathbf{x}} - ma \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{p}} = 0$ $\Rightarrow$ infinite hierarchy

continuity eq. $\frac{\partial \delta}{\partial \tau} + \frac{\partial}{\partial x_i} [(1+\delta) v_i] = 0$

Euler eq. $\frac{\partial v_j}{\partial \tau} + \mathcal{H} v_j + v_i \frac{\partial v_j}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} - \frac{1}{\rho} \frac{\partial}{\partial x_i} (\rho \sigma_{ij})$

....

2) Numerical simulations at late times (non-linear regime)

- Neglect the late-time velocity dispersion

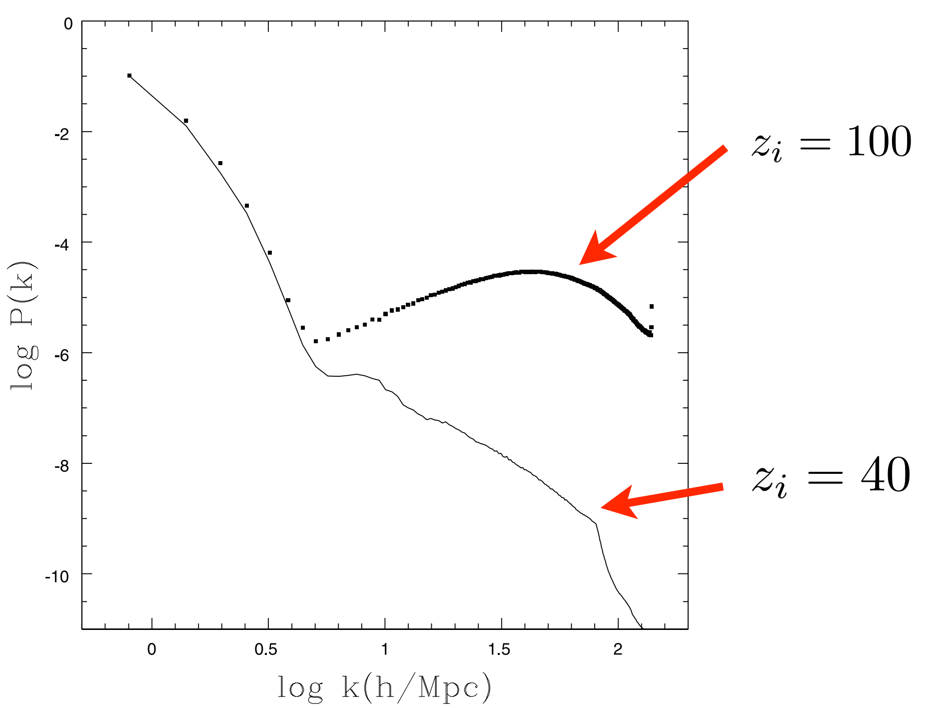
→ the only difference from CDM is the high-k cutoff of the "initial" power spectrum at $z_i \sim 50 - 100$

- Add a random velocity to each "macro-particle" $(M \sim 10^5 M_\odot)$

→ mimic the upper bound on the coarse-grained *f(x,p)* (Liouville theorem)

However, this can lead to spurious effects: gives rise to an high-k tail in the initial power spectrum

White-noise velocity field → $P(k) \propto k^2$



$z = 40$

Colin et al. (2008)

Including the effects of a late-time ($z<100$) velocity dispersion is not so easy, in either the analytic or numerical approaches.

→ Estimate the impact of a late-time velocity dispersion, to check that it can be neglected on large scales (outside of halos).

→ Estimate the sensitivity on the "initial" redshift of the simulations.

C- Effective Euler equation

From the linearized Vlasov equation one can derive:

Boyanovsky et al.(2008), Boyanovsky & Wu (2011)

$$\frac{\partial^2\tilde{\delta}}{\partial\tau^2} + \mathcal{H}\frac{\partial\tilde{\delta}}{\partial\tau} - \left(\frac{3}{2}\Omega_m\mathcal{H}^2 - k^2c_s^2\right)\tilde{\delta} = S[\tilde{\delta},\tau]$$

~ sound speed

free-streaming wavenumber ~ Jeans wavenumber $\quad k_{fs}^2 = \dfrac{3\Omega_m\mathcal{H}^2}{2c_s^2}$

- However, there is no thermodynamical pressure

$$k_{fs}(\tau) \propto \sqrt{a(\tau)} \qquad\qquad k_{fs}(\tau_{\rm eq}) \simeq \frac{11.17}{\sqrt{\bar{y}^2}}\left(\frac{m}{1\mathrm{keV}}\right)\left(\frac{g_d}{2}\right)^{1/2}\mathrm{Mpc}^{-1}$$

- $S = S_{\rm NB} + S_{\rm B}$

inhomogeneous term (integral over the "initial" distribution at matter-rad. eq.), subdominant/growing mode

memory term (integral over the past) $\quad \propto (k/k_{fs})^4$ at low k

We consider the CDM-like hydrodynamical equations, with a new pressure-like term

$$\begin{cases} \dfrac{\partial \delta}{\partial \tau} + \nabla \cdot [(1+\delta)\mathbf{v}] = 0 \\[2ex] \dfrac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\Phi - c_s^2 \dfrac{\nabla \rho}{\bar{\rho}} \\[2ex] \nabla^2 \Phi = \dfrac{3}{2}\Omega_m \mathcal{H}^2 \delta \end{cases}$$

- Simplest closure that agrees with the Vlasov analysis at linear order (with S=0)

- We use for c_s the result from the linearized Vlasov eq.

- $\dfrac{\nabla\rho}{\bar{\rho}} = \nabla\delta$ is better behaved than $\dfrac{\nabla\delta}{1+\delta} = (1-\delta+\ldots+(-1)^p\delta^p+..)\nabla\delta$

  always mimics the slow-down of gravitational collapse due to the non-zero velocity dispersion

We use as initial conditions, at redshift $z_i = 100$

the linear power spectrum with the WDM high-k cutoff :

$$P_{L,\mathrm{WDM}}(k, z=0) = P_{L,\mathrm{CDM}}(k, z=0)\, T(k)^2$$

Viel et al.(2005)

$$T(k) = [1 + (\alpha k)^{2\nu}]^{-5/\nu}$$

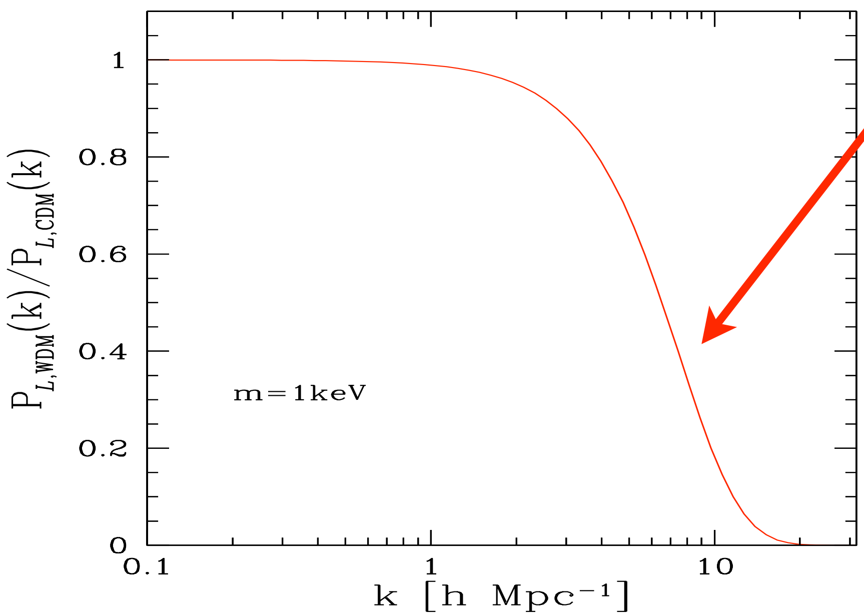
$$\alpha = 0.049 \left(\frac{m}{1\mathrm{keV}}\right)^{-1.11} \left(\frac{\Omega_m}{0.25}\right)^{0.11} \left(\frac{h}{0.7}\right)^{1.22} h^{-1}\mathrm{Mpc} \qquad \nu = 1.12$$

→ We can compare the impact of the late-time velocity dispersion (z<100) with the impact of the high-k cutoff (early free-streaming).

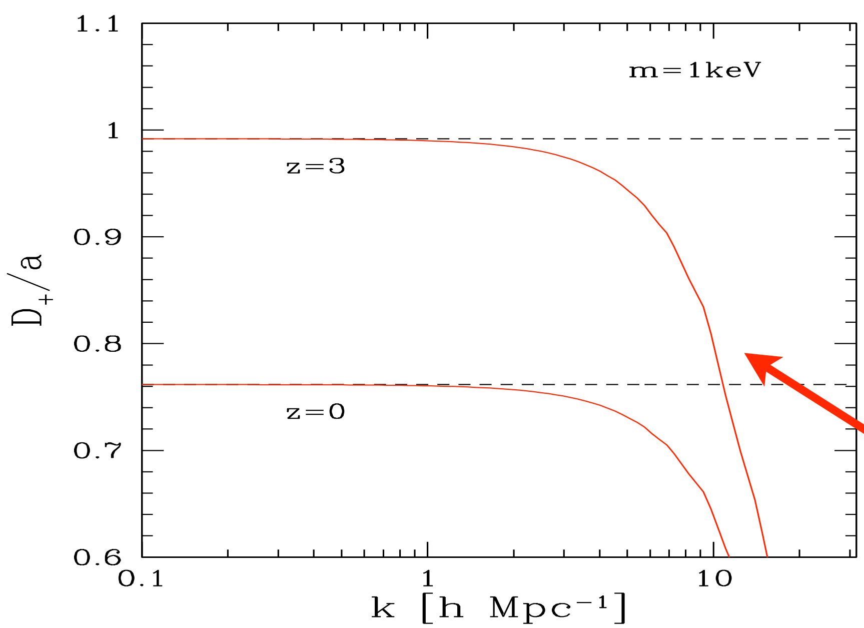
D- Results

Because of the "pressure-like" term $-k^2 c_s^2$ the linear growing and decaying modes depend on k .

high-k cutoff due to early free-streaming



ratio of the WDM/CDM
linear power spectra



linear growing mode $D_+(k,z)$

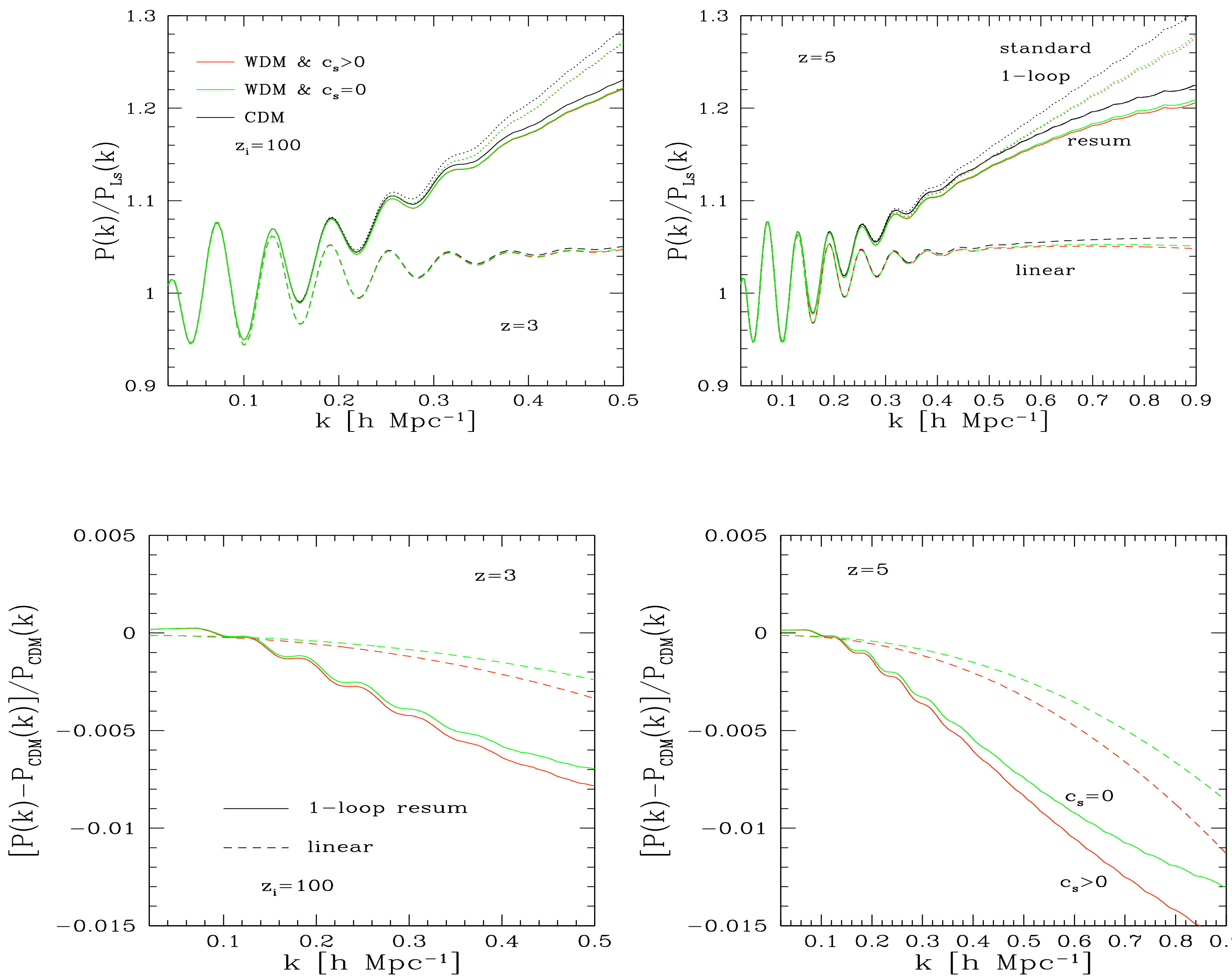
(normalized "initially" to: $D_+(k,z_i) = D_{+,\mathrm{CDM}}(z_i)$)

slower linear growth at high k since z=100

One-loop power spectrum (on perturbative scales)

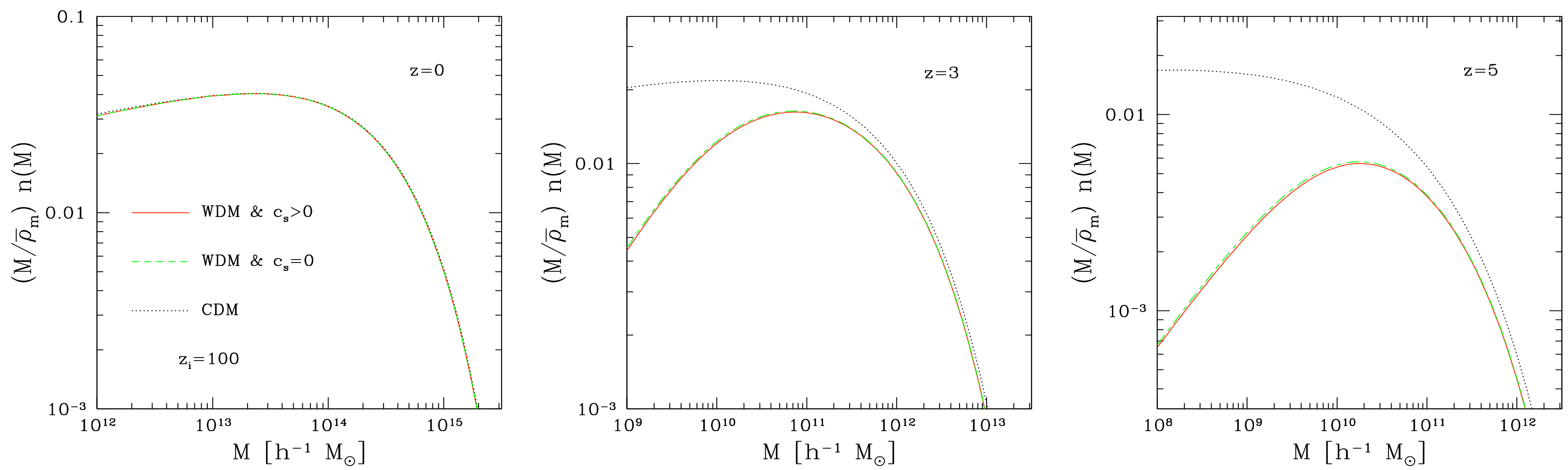
power spectrum:

relative deviation from CDM:



- relative deviation from CDM on perturbative scales is very small (1%)
- below the accuracy of standard perturbation theory
- the effect of the late-time velocity dispersion is negligible

Halo mass function



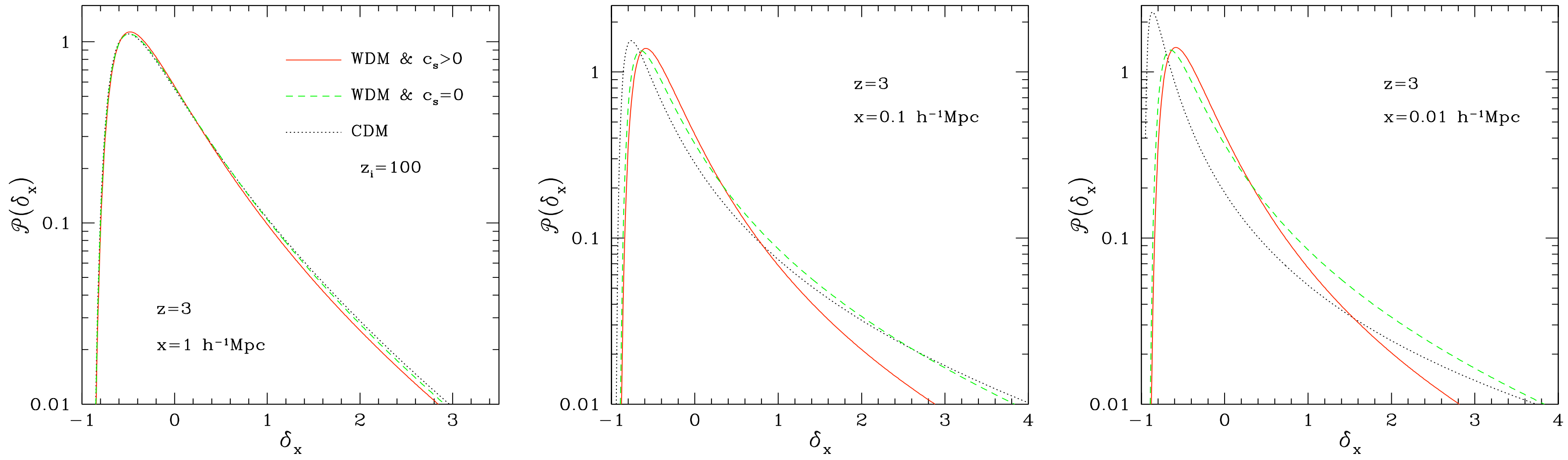
Press-Schechter scaling:

$$n(M)\frac{dM}{M} = \frac{\bar{\rho}_m}{M} f(\nu) \frac{d\nu}{\nu} \qquad \nu = \frac{\delta_c(M)}{\sigma_q} \qquad \delta_c(M) = \mathcal{F}_q^{-1}(200)$$

Because of the pressure-like term in the equations of motion, the spherical dynamics and the threshold δ_c depend on scale, whence on mass.

- the effect of the late-time velocity dispersion is negligible (at z<=5)

Probability distribution of the density contrast within spherical cells



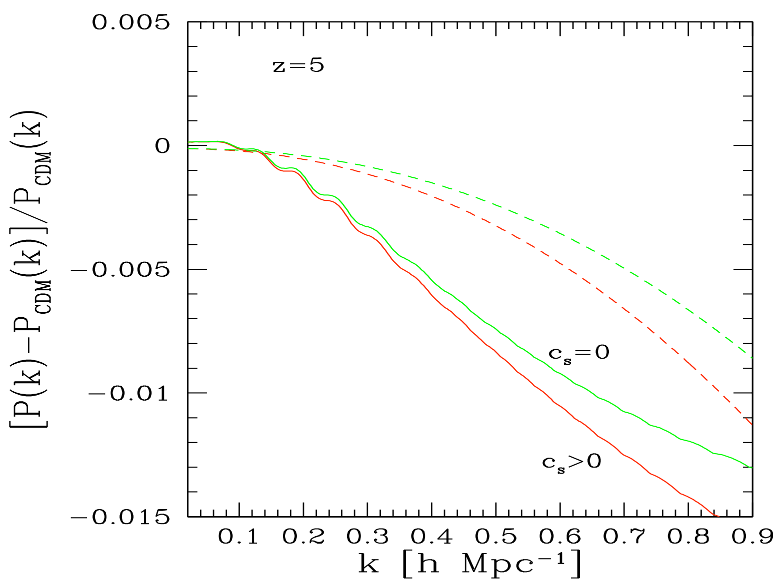
- WDM leads to a less advanced stage of non-linear evolution

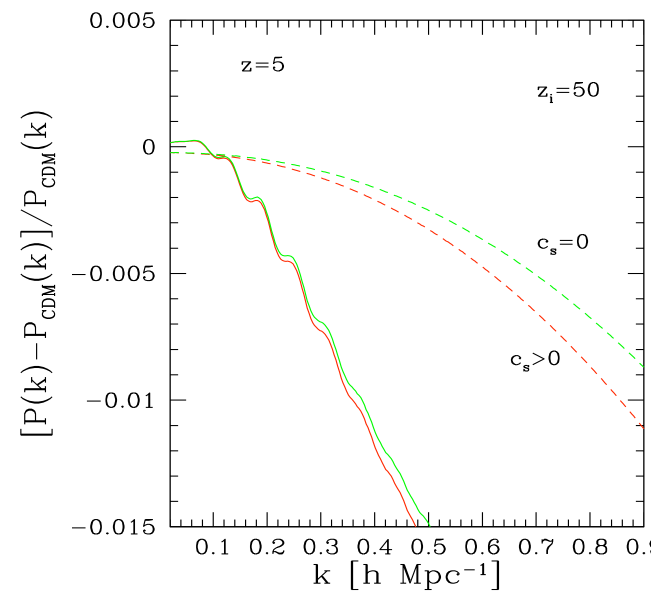
- the late-time velocity dispersion has a non-negligible effect on the PDF

- this may have a small impact on the PDF of the Lyman-alpha forest absorption lines

Impact of low initial redshift

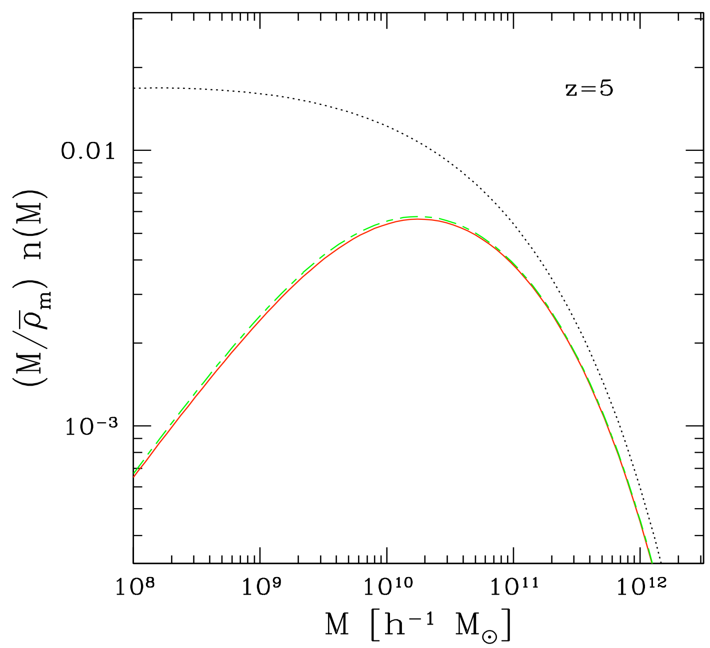
$z_i = 100$ | $z_i = 50$

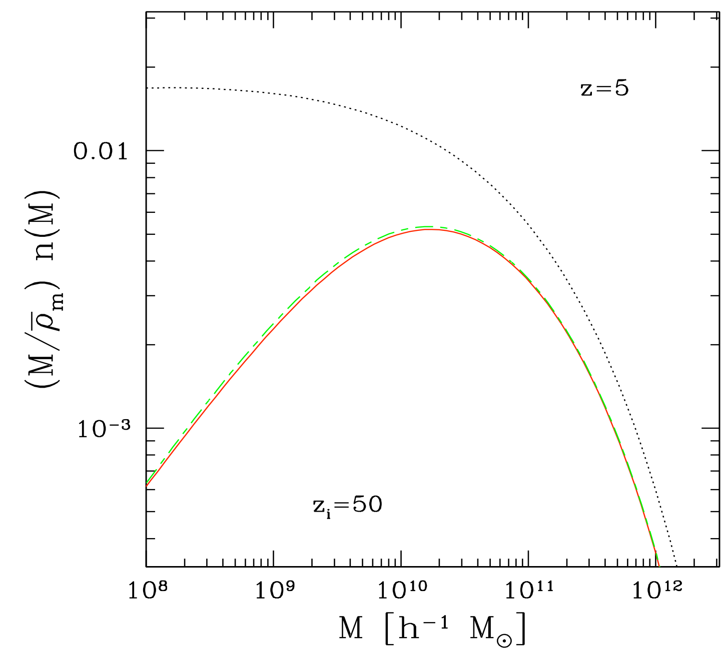




power spectrum

- underestimation of the power spectrum on perturbative scales

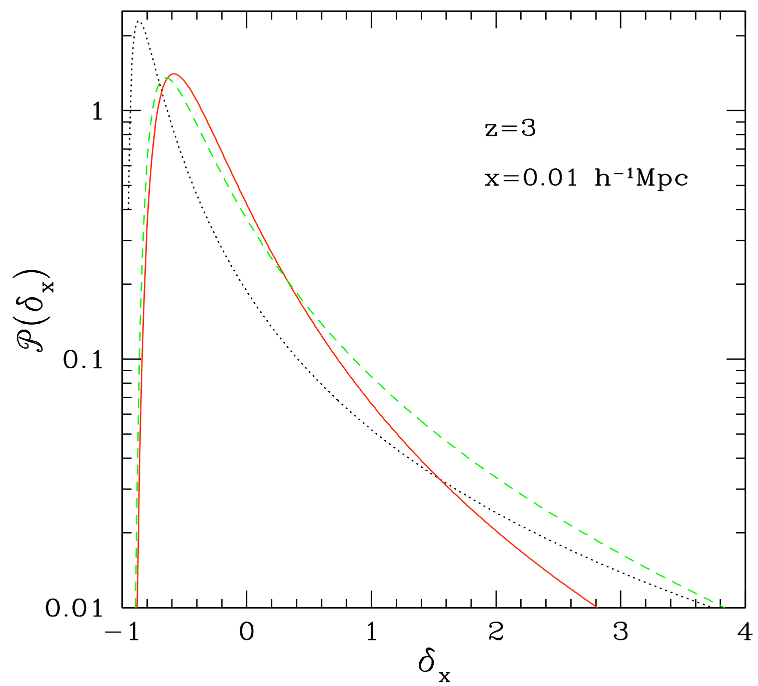




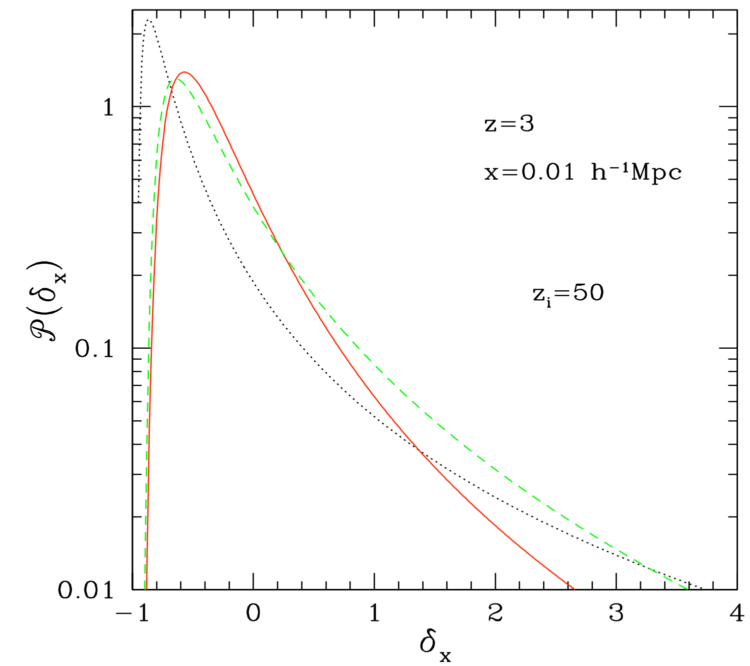
mass function

- underestimation of the large-mass tail of the halo mass function

$$z_i = 100$$



$$z_i = 50$$



- rather small effect on the PDF of the density contrast on the scales associated with Lyman-alpha cloud at z=3

E- Conclusion

- For most practical purposes, the late-time velocity dispersion has a negligible effect on large-scale structures (outside of virialized halos), in particular for the power spectrum on perturbative scales and the halo mass function (at least at low z).

- There is a small effect on the PDF of the density contrast on scales associated with Lyman-alpha clouds.

- Using a low initial redshift, z~50 (with linear theory initial conditions), significantly underestimates the power spectrum on perturbative scales and the large-mass tail of the halo mass function.

- On large scales (cosmic web) one can use the tools used for CDM.

- It is possible to include a pressure-like term, which ensures consistency with the Vlasov linear analysis (with S=0).

Perturbative methods

Scales of interest:

$$x > 10h^{-1}\mathrm{Mpc}$$

$$k < 0.4h\mathrm{Mpc}^{-1}$$

Linear to weakly nonlinear regime

Future observations require a percent-level accuracy for theoretical predictions

F. Bernardeau & P.V., 2008 - P.V., 2004; 2007a,b; 2008; 2010a - P.V. & T. Nishimichi, 2011 - Ph. Brax & P.V. 2012

F. Bernardeau, M. Crocce, E. Sefusatti; 2010 - M. Crocce & R. Scoccimarro, 2006a,b; 2008 - S. Matarrese & M. Pietroni, 2007 - M. Pietroni, 2008 - A. Taruya & T. Hiramatsu, 2008

A- Standard perturbation theory

1) Hydrodynamical approximation (single-stream approximation)

(C)DM+baryons: (pressure-less) & irrotational perfect fluid

$$\frac{\partial\delta}{\partial\tau} + \nabla\cdot[(1+\delta)\mathbf{v}] = 0$$

$$\frac{\partial\mathbf{v}}{\partial\tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla\Phi - c_s^2\frac{\nabla\rho}{\bar\rho}$$

$$\Delta\phi = \frac{3}{2}\Omega_{\rm m}\mathcal{H}^2\delta$$

density contrast:

$$\delta(\mathbf{x},t) = \frac{\rho(\mathbf{x},t)-\bar\rho}{\bar\rho}$$

2) Solution as a perturbative expansion over powers of the linear mode

$$\tilde\delta(\mathbf{k},\tau) = \sum_{n=1}^{\infty}\tilde\delta^{(n)}(\mathbf{k},\tau) \qquad \text{with} \qquad \tilde\delta^{(n)} \propto (\tilde\delta_L)^n$$

3) Gaussian average for statistical quantities

$$C_2 = \langle\delta\delta\rangle = \langle\delta^{(1)}\delta^{(1)}\rangle + \langle\delta^{(3)}\delta^{(1)}\rangle + \langle\delta^{(1)}\delta^{(3)}\rangle + \langle\delta^{(2)}\delta^{(2)}\rangle + \ldots$$

Using the Poisson equation, one obtains 2 equations for the density and velocity fields, which are quadratic.

Introduce the velocity divergence: $\theta = \nabla \cdot \mathbf{v}$

a 2-component vector: $\psi = \begin{pmatrix} \delta \\ -\theta/\dot{a} \end{pmatrix}$

Quadratic equation of motion, with a linear operator that may depend on wavenumber (pressure-like term or modified gravity):

$$\mathcal{O} \cdot \tilde{\psi} = K_s \cdot \tilde{\psi}\tilde{\psi} \qquad \mathcal{O} = \begin{pmatrix} \frac{\partial}{\partial \ln a} & -1 \\ -\frac{3\Omega_m}{2}[1+\epsilon(k,a)] & \frac{\partial}{\partial \ln a} + \frac{1-3w\Omega_{\rm de}}{2} \end{pmatrix}$$

WDM pressure-like term: $\epsilon(k,a) = -\dfrac{k^2}{k_{fs}(a)^2}$

Modified gravity: $\epsilon(k,a) = \dfrac{2\beta(a)^2 k^2}{k^2 + m(a)^2 a^2}$

$$\mathcal{O} \cdot \tilde{\psi} = K_s \cdot \tilde{\psi}\tilde{\psi}$$

Standard perturbative expansion: $\tilde{\psi}(x) = \sum_{n=1}^{\infty} \tilde{\psi}^{(n)}(x)$ with $\tilde{\psi}^{(n)} \propto (\tilde{\psi}_L)^n$

solved by recursion up to the required order *n* :

$$\mathcal{O} \cdot \tilde{\psi}^{(n)} = K_s(x; x_1, x_2) \sum_{\ell=1}^{n-1} \tilde{\psi}^{(\ell)}(x_1)\tilde{\psi}^{(n-\ell)}(x_2)$$

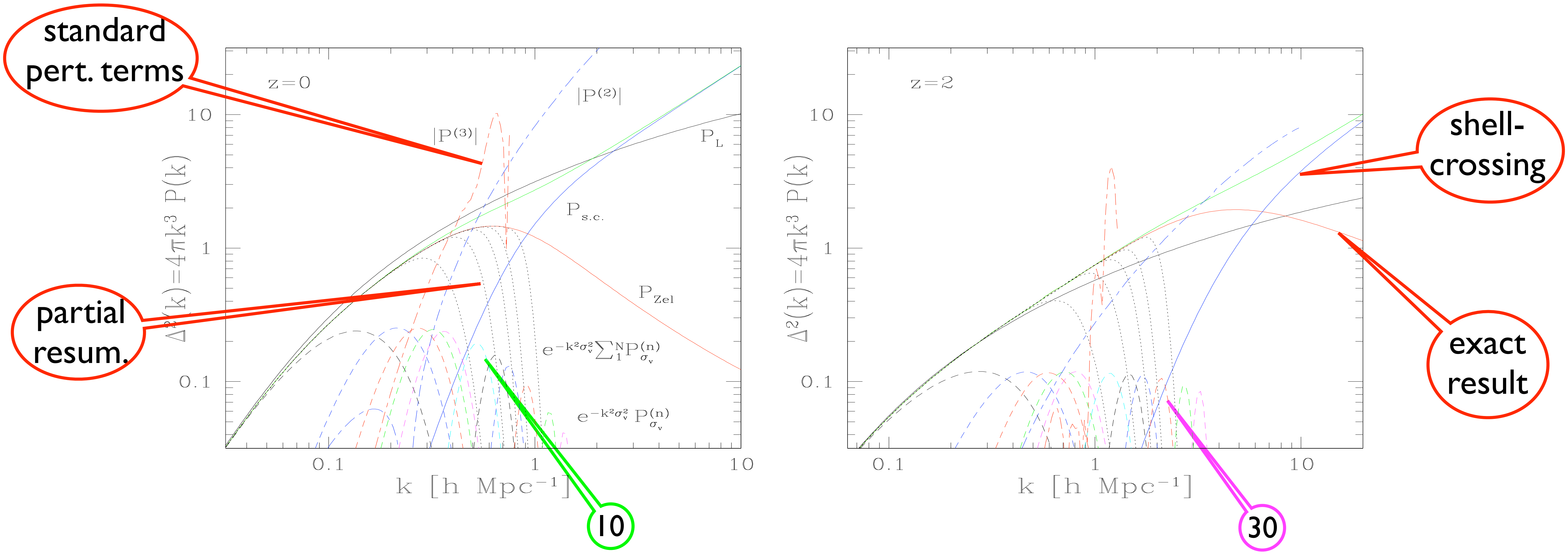
$$\tilde{\delta}(\mathbf{k}, a) = \sum_{n=1}^{\infty} \int d\mathbf{k}_1..d\mathbf{k}_n \, \delta_D(\mathbf{k}_1 + .. + \mathbf{k}_n - \mathbf{k}) \, F_n^s(\mathbf{k}_1, .., \mathbf{k}_n; a) \, \tilde{\delta}_{L0}(\mathbf{k}_1)..\tilde{\delta}_{L0}(\mathbf{k}_n)$$

If $\epsilon(k, a)$ depends on wavenumber the time-dependence of F_n^s does not factor out.

$$P(k) = \langle \delta\delta \rangle = \langle \delta^{(1)}\delta^{(1)} \rangle + \langle \delta^{(3)}\delta^{(1)} \rangle + \langle \delta^{(1)}\delta^{(3)} \rangle + \langle \delta^{(2)}\delta^{(2)} \rangle + ...$$

Test on the Zeldovich Dynamics

(particles moving on straight lines according to their initial velocity)



- standard perturbation theory is not well-behaved

- many orders are relevant before the nonperturbative term (shell crossing) dominates

Need for improved (resummation) schemes

B- Path-integral formulation

1) Generating functional of many-body correlations

$$\psi = \begin{pmatrix} \delta \\ -\theta/\dot{a} \end{pmatrix}$$

$$\boxed{Z[j] = \langle e^{j\cdot\psi} \rangle} = \int \mathcal{D}\mu_I \; e^{j\cdot\psi[\mu_I] - \frac{1}{2}\mu_I\cdot\Delta_I^{-1}\cdot\mu_I}$$

eq. of motion

Gaussian init. cond.

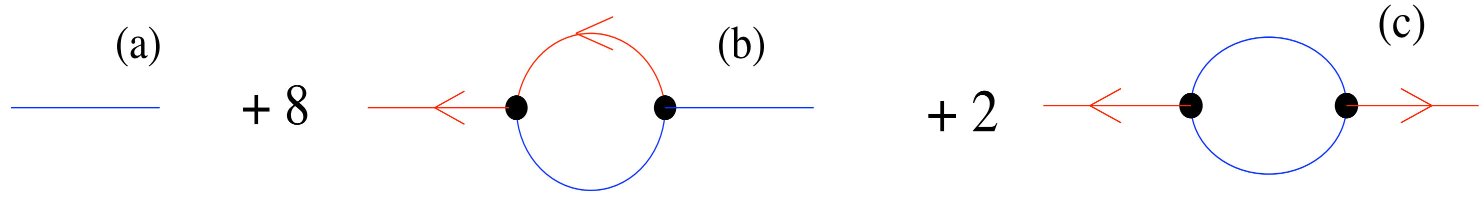
$$Z[j] = \int \mathcal{D}\psi\, \mathcal{D}\lambda \; e^{j\cdot\psi + \lambda\cdot(-\mathcal{O}\cdot\psi + K_s\cdot\psi\psi) + \frac{1}{2}\lambda\cdot\Delta_I\cdot\lambda}$$

2) Use perturbative schemes to compute Z[j]

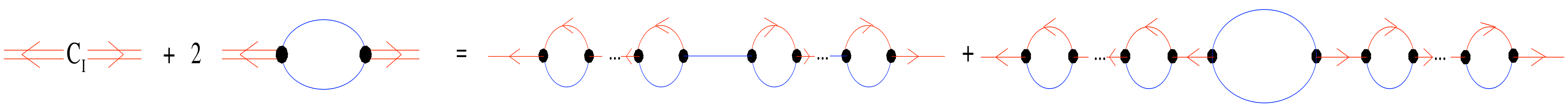
$\Rightarrow$ Power spectrum, bispectrum, ...

1-loop order
(i.e., up to)

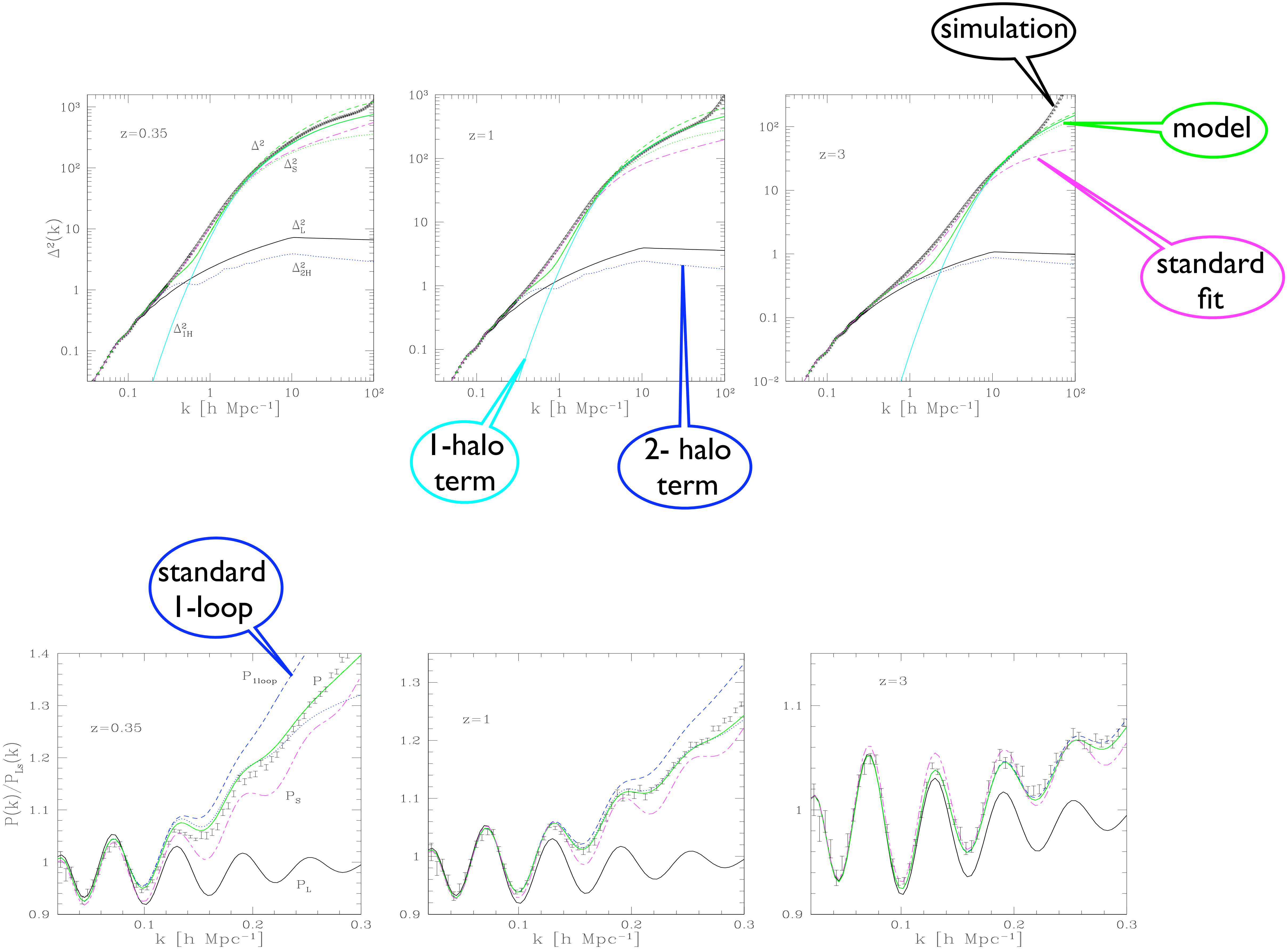
a) Standard perturbation theory

$P(k) = C_2 =$ 

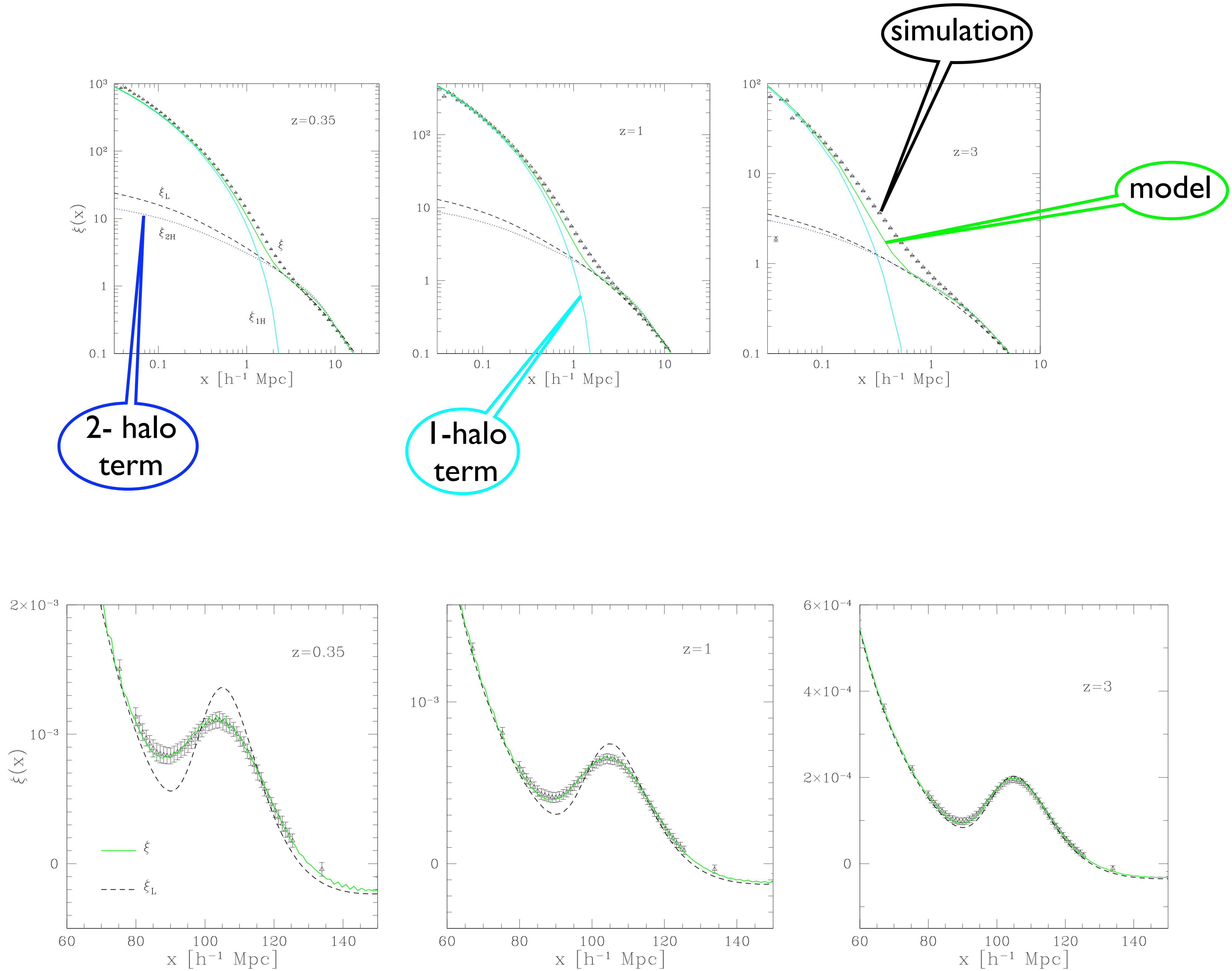
b) Direct steepest-descent method

$P(k) =$ 

Results for the power spectrum (CDM)



Results for the two-point correlation function (CDM)



Higher-order statistics: the bispectrum

In order to break degeneracies, it is useful to consider higher-order statistics beyond the power spectrum (i.e., 2-pt correlation).

↓

3-pt correlation / bispectrum

This is also useful to constrain primordial non-Gaussianities

P.V. & T. Nishimichi, 2011

$$\langle \tilde{\delta}(\mathbf{k}_1)\tilde{\delta}(\mathbf{k}_2)\tilde{\delta}(\mathbf{k}_3)\rangle = \delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\, B(k_1,k_2,k_3)$$

As in the halo model (but from a Lagrangian point of view), we decompose the bispectrum as

$$B = B_{1\mathrm{H}} + B_{2\mathrm{H}} + B_{3\mathrm{H}}$$

"1- halo" and "2-halo" terms

nonperturbative contributions

"3-halo term"

perturbative contribution

$$B_{3\mathrm{H}} = B_{\mathrm{pert}}$$

$$B_{1\mathrm{H}} = \int \frac{d\nu}{\nu}\, f(\nu) \left(\frac{M}{\bar{\rho}(2\pi)^3}\right)^2 \prod_{j=1}^{3} \left(\tilde{u}_M(k_j) - \tilde{W}(k_j\, q_M)\right)$$

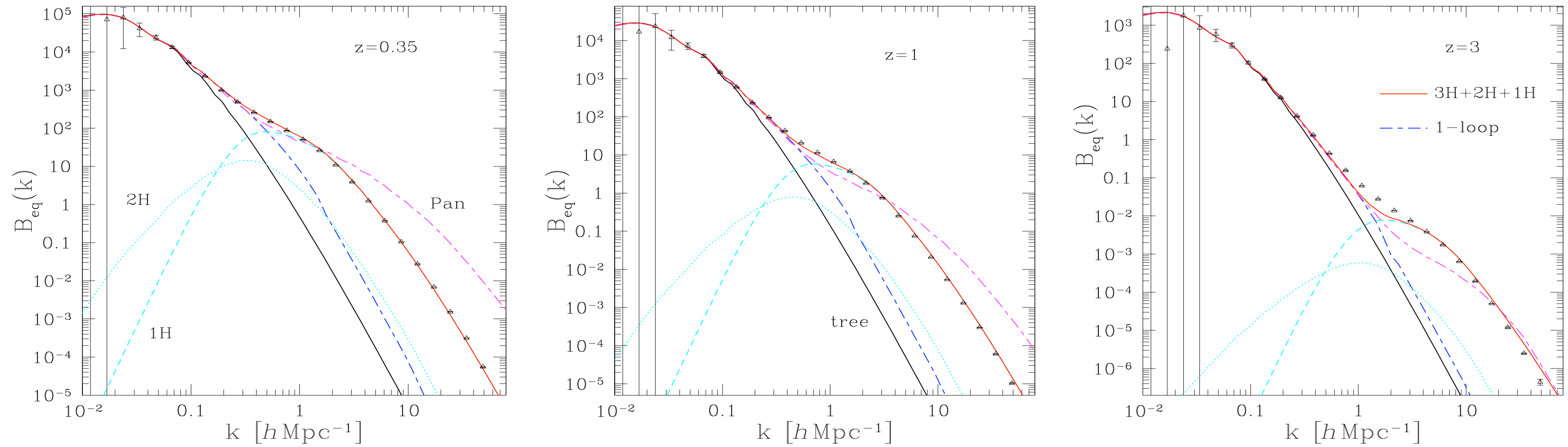
$$B_{2\mathrm{H}} = P_L(k_1) \int \frac{d\nu}{\nu}\, f(\nu)\, \frac{M}{\bar{\rho}(2\pi)^3} \prod_{j=2}^{3} \left(\tilde{u}_M(k_j) - \tilde{W}(k_j\, q_M)\right) + 2\,\mathrm{cyc.}$$

counterterms

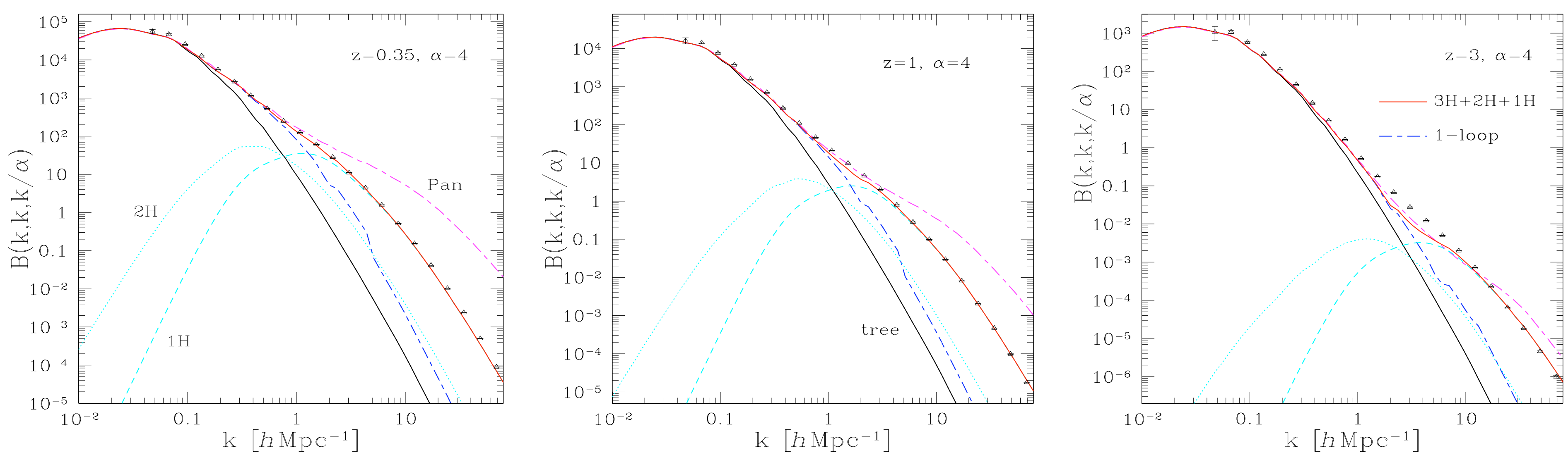
$$B_{1\mathrm{H}} \propto k_j^2 \quad \text{for} \quad k_j \to 0$$

$$B_{2\mathrm{H}} \propto P_L(k_j) \quad \text{for} \quad k_j \to 0$$

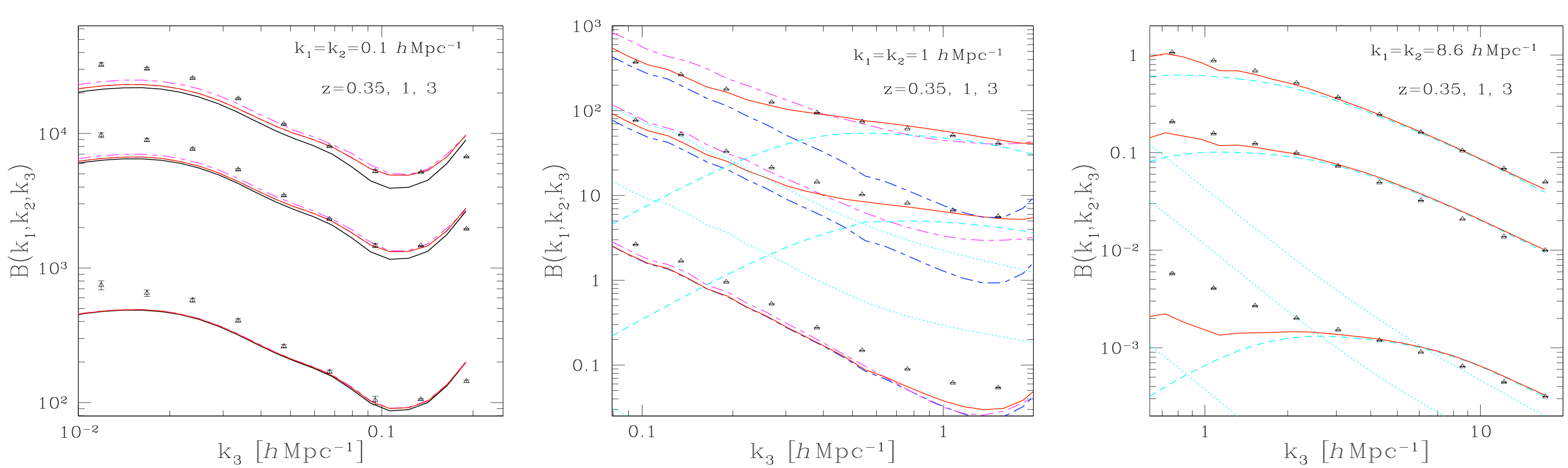
Equilateral triangles



Isosceles triangles at fixed length ratio = 4



Isosceles triangles at fixed equal-sides length



Spherical collapse

In CDM, before shell-crossing, all shells move independently.

If $\epsilon(k, a)$ depends on wavenumber, all shells are coupled.

$$\ddot{r} = -\frac{4\pi\mathcal{G}}{3} r \left[\rho_m(< r) + (1+3w)\bar{\rho}_{de} + \bar{\rho}_m \int_0^\infty dk\, 4\pi k^2 \epsilon(k)\, \tilde{\delta}(k)\, \tilde{W}(kx) \right]$$

dependence on the full density profile

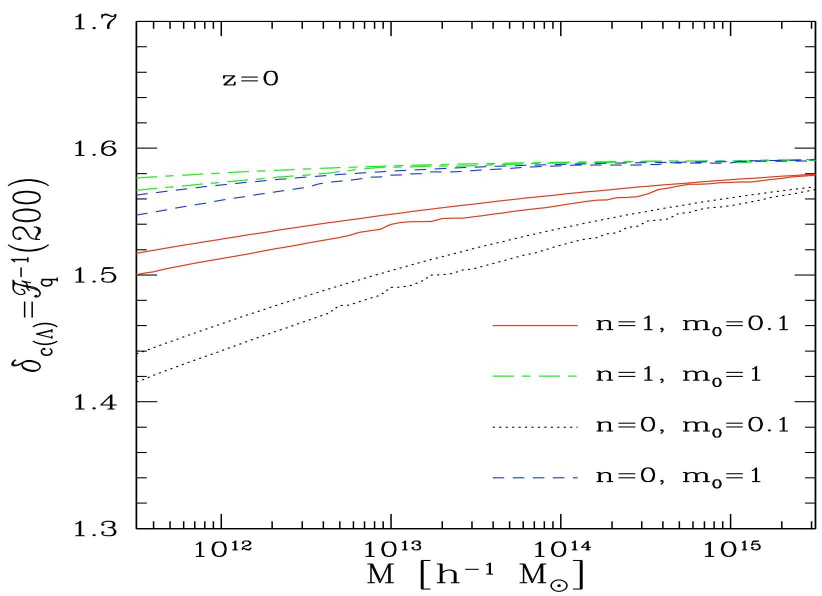
A simple approximation: use a typical profile parameterized by the shell of interest:

$$\delta(\mathbf{x}) = \frac{\delta_{x_M}}{\sigma_{x_M}^2} \int_{V_M} \frac{d\mathbf{x}'}{V_M}\, C_{\delta_L \delta_L}(\mathbf{x}, \mathbf{x}')$$

$$\tilde{\delta}(k) = \frac{\delta_{x_M}}{\sigma_{x_M}^2}\, P_L(k)\, \tilde{W}(k x_M)$$

This implies that the dynamics of collapse depends on scale (or mass).

The linear density threshold δ_c to reach a density contrast of 200 depends on mass.



case where $\epsilon(k, a) > 0$

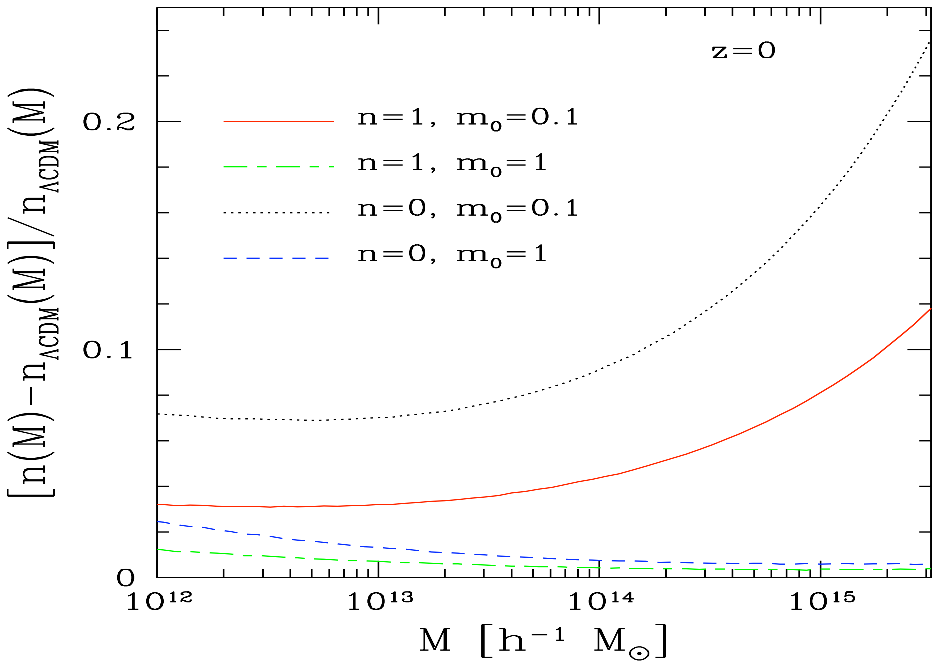
(mod. grav.)

Halo mass function

$$M \to \infty : \quad \ln[n(M)] \sim -\frac{\delta_c(M)^2}{2\sigma(M)^2}$$ with $$\delta_c(M) = \mathcal{F}_q^{-1}(200)$$

Use the Press-Schechter scaling:

$$n(M)\frac{dM}{M} = \frac{\bar{\rho}_m}{M}\, f(\nu)\,\frac{d\nu}{\nu}$$ with $$\nu = \frac{\delta_c(M)}{\sigma(M)}$$



Relative deviation of the mass function

case where $\epsilon(k,a) > 0$

(mod. grav.)

Probability distribution of the density contrast

From the spherical dynamics we can also obtain the PDF of the density contrast within spherical cells, in the weakly non-linear regime.

Introduce the cumulant generating function (Laplace transform):

$$e^{-\varphi(y)/\sigma_x^2} \equiv \langle e^{-y\delta_x/\sigma_x^2} \rangle = \int_{-1}^{\infty} d\delta_x \; e^{-y\delta_x/\sigma_x^2} \; \mathcal{P}(\delta_x)$$

$$e^{-\varphi(y)/\sigma_x^2} = (\det C_{\delta_L\delta_L}^{-1})^{1/2} \int \mathcal{D}\delta_L \; e^{-S[\delta_L]/\sigma_x^2}$$

where

$$S[\delta_L] = y\,\delta_x[\delta_L] + \frac{\sigma_x^2}{2}\,\delta_L \cdot C_{\delta_L\delta_L}^{-1} \cdot \delta_L$$

On large scales, we obtain: $\sigma_x \rightarrow 0: \quad \varphi(y) \rightarrow \min_{\delta_L} S[\delta_L]$

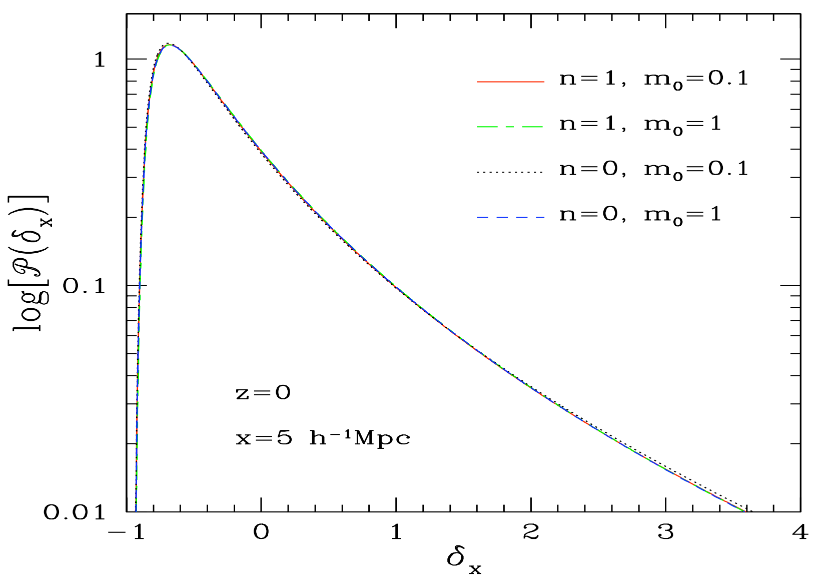
For spherical cells, we can look for the spherical minimum (saddle-point)

In GR the radial profile is given by: $\delta_{Lq'} = \delta_{Lq} \frac{\sigma^2_{q,q'}}{\sigma^2_q}$

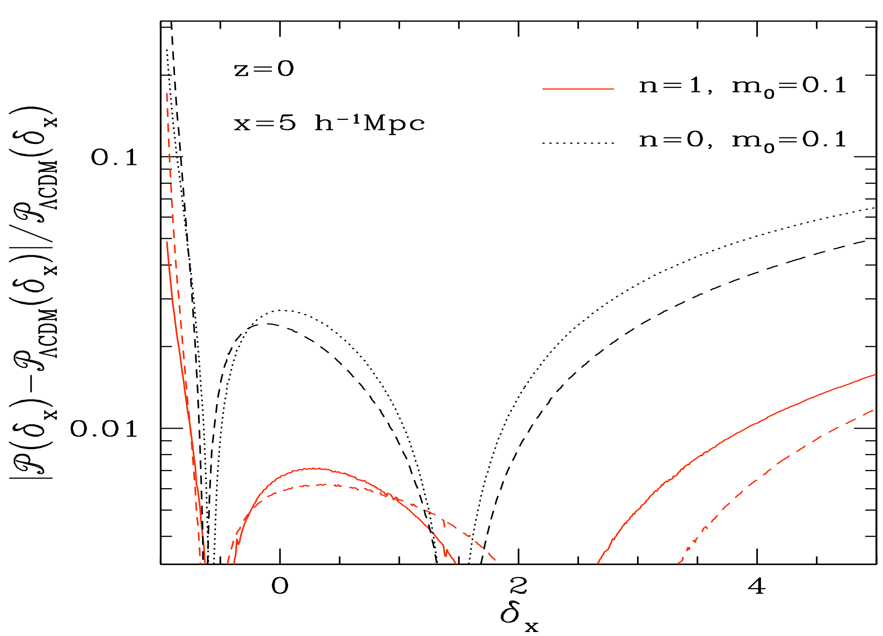
and the Lagrangian radius *q* associated with the Eulerian radius *x* is given by:

$$\begin{cases} q^3 = (1+\delta_x)\, x^3 \\ \\ \delta_x = \mathcal{F}_q(\delta_{Lq}) \end{cases}$$

This determines the Laplace transform $\varphi(y)$, whence the PDF $\mathcal{P}(\delta_x)$.



Probability distribution function



Relative deviation

case where $\epsilon(k,a) > 0$

(mod. grav.)

Conclusion

Theoretical approach that is complementary to numerical simulations and observations.

- explore regimes that are beyond the reach of numerical simulations (e.g., very large scales)
- understand the main properties of the dynamics (Burgers)

Prospects

- find out which resummation schemes are the most accurate/efficient ?
- improve the modeling of the transition scales
- further develop Lagrangian approaches (useful for redshift-space distortions)
- improve the generalization to warm dark matter ? more complex components (clustering quintessence,...)
- going beyond the fluid approximation: phase-space description ?