Statistics of Dark Matter Halos from the Excursion Set Approach

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based upon Lapi+13, ApJ, in press [preprint arXiv:1305.7382]

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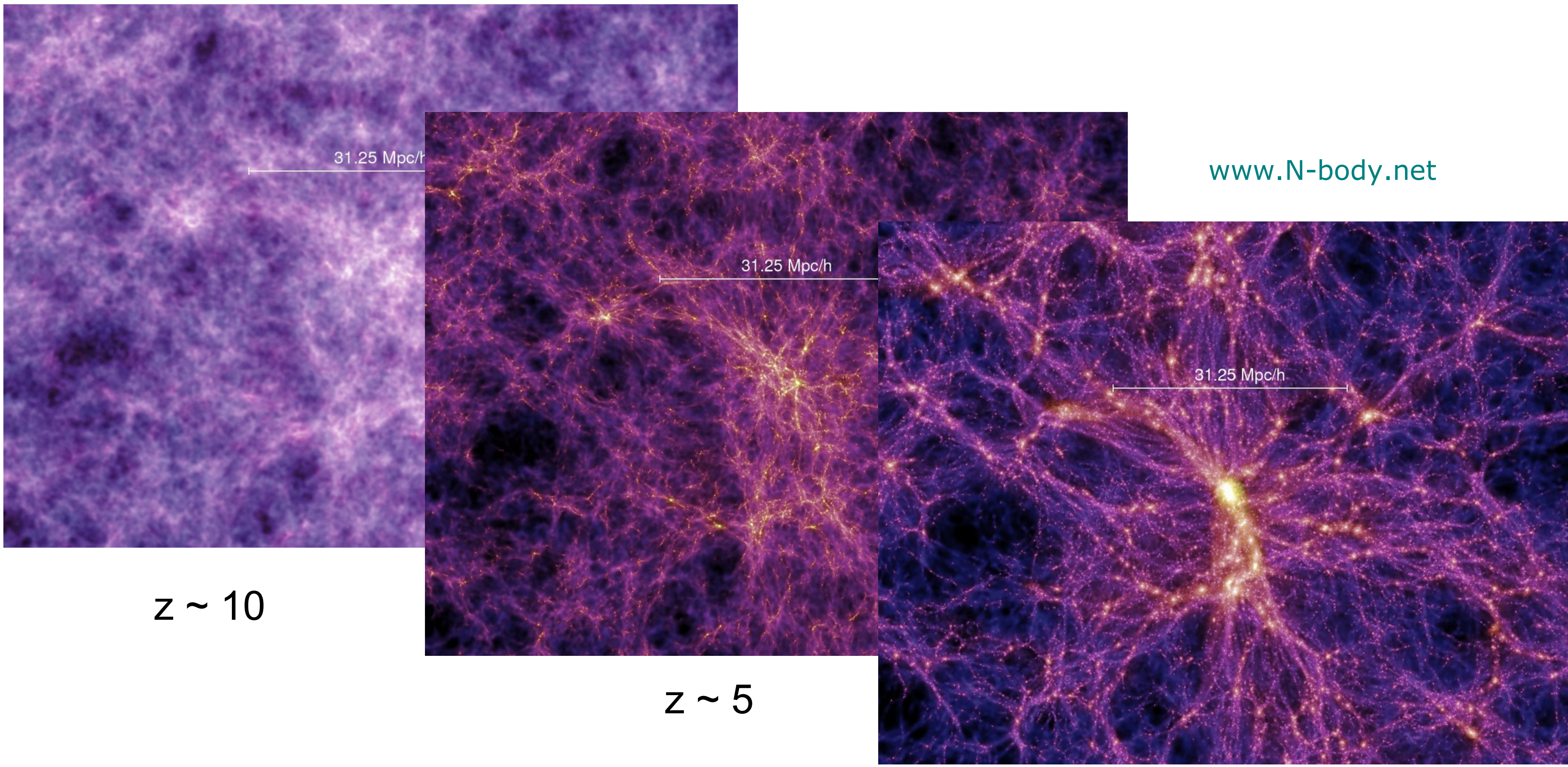
Introduction

- Halos form from initial DM density perturbations grown by gravitational instability, then collapsed and virialized under self-gravity → N-body simulations



www.N-body.net

z ~ 10

z ~ 5

z ~ 0

Introduction

- Halo abundance as a function of mass and redshift

  → Halo Mass Function

$$\frac{dN}{dM}(M, z)$$

Many fitting functions of simulation outcomes at >10% precision, but:

→ empirical parameters

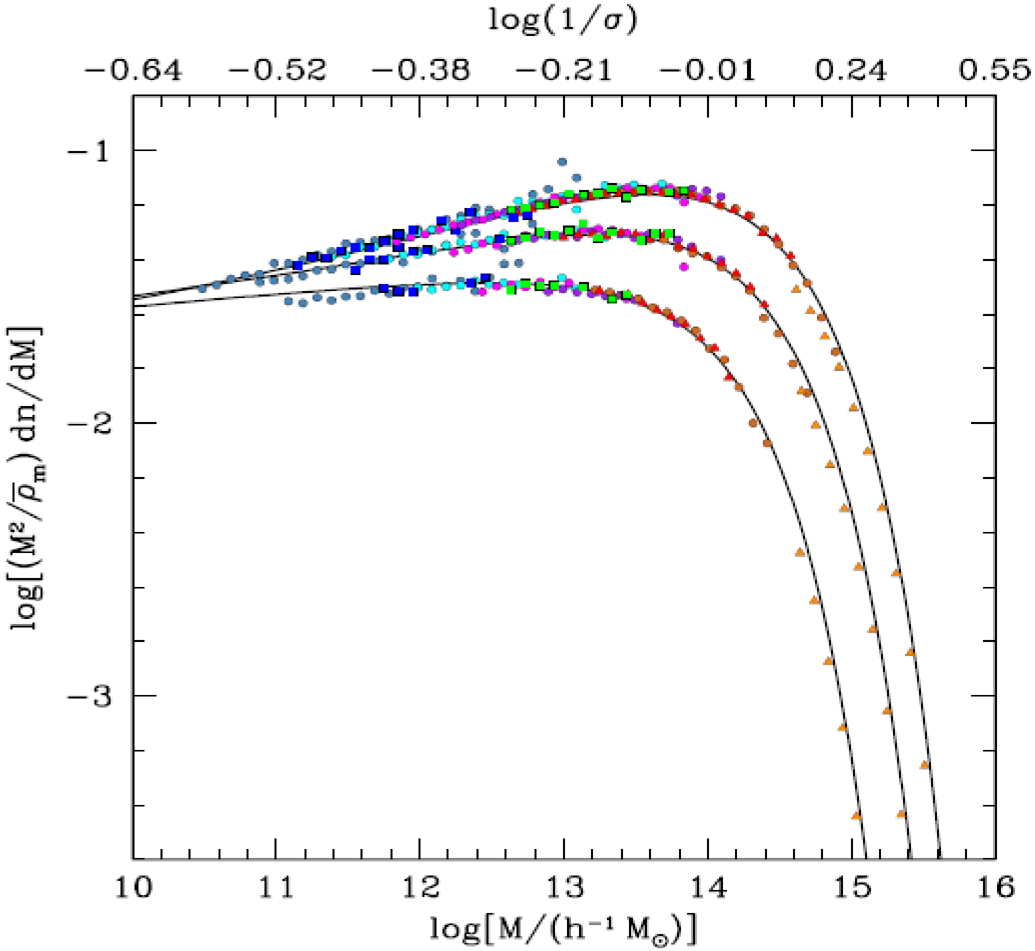
→ dependent on sim setup

→ poor flexibility

→ complex expressions

Jenkins+01, Springel+05, Warren+06, Tinker+08, Crocce+10, Angulo+12, Watson+13, many others

Tinker+08

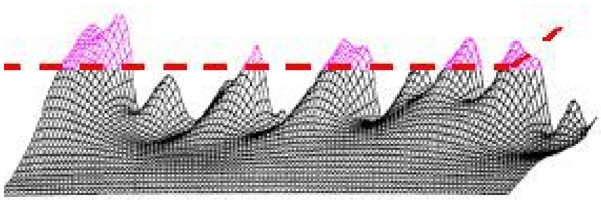


Excursion Set Approach

- Theoretical framework → Excursion Set Approach

Press&Schechter74, Bond+01, Lacey&Cole93

Density contrast $\delta(\mathbf{x}) \equiv \rho(\mathbf{x})/\bar{\rho} - 1$



spatially smoothed $\delta(R) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \hat{W}_R(k)\, \hat{\delta}(\mathbf{k})$

depends on power spectrum of fluctuations $\langle \hat{\delta}(\mathbf{k})\hat{\delta}(\mathbf{k}')\rangle = (2\pi)^3\, P(k)\, \delta_D(\mathbf{k}+\mathbf{k}')$

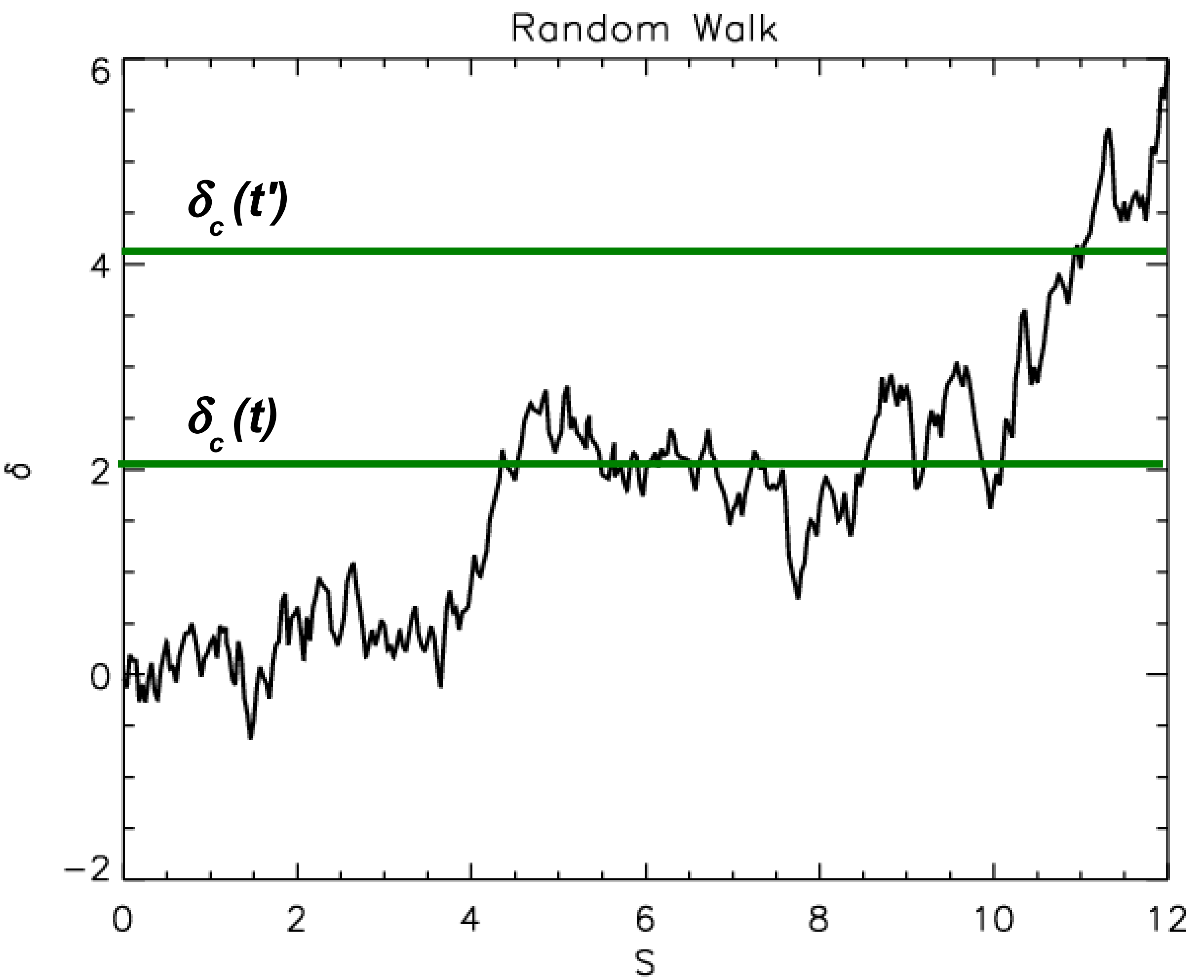
performs a stochastic motion $\frac{\partial \delta(S)}{\partial S} = \eta(S)$ $\langle \eta(S_1)\eta(S_2)\rangle = \delta(S_1 - S_2)$

under a Gaussian noise with variance $S(R) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\, \hat{W}_R^2(k)\, P(k)$

If k-sharp filter W_R adopted then random walk is Markovian (uncorrelated steps)

Excursion Set Approach

Time *t*

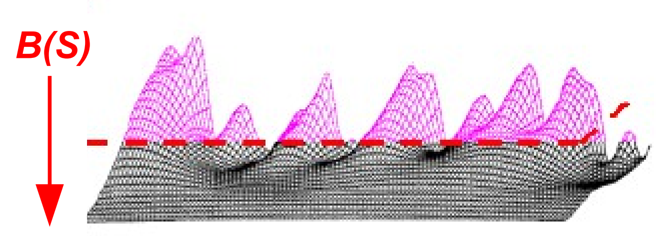


Scale *R*, Mass *M*

Excursion Set Approach

- Mass function is linked to the distribution *f(S)* for the walk to first cross a given barrier *B(S)*

$$\frac{dN}{dM} = \frac{\bar{\rho}}{M^2} \left| \frac{d\log S}{d\log M} \right| S f(S)$$



other crossings would correspond to smaller scales which will be eaten up by the largest, not to be overcounted ("cloud-in-cloud" problem).

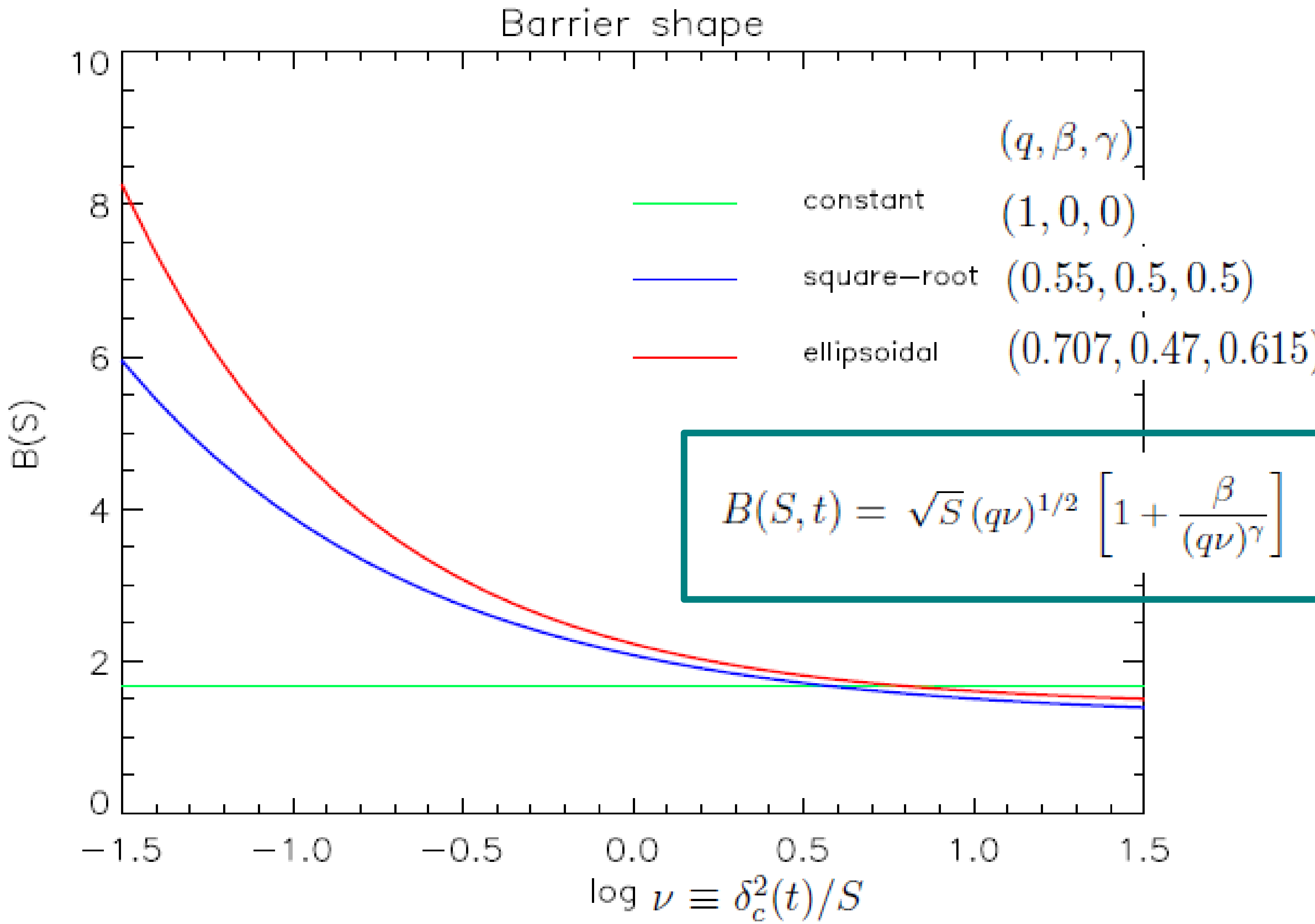
The barrier *B(S)* includes the time-dependent threshold for collapse $\delta_c(t)$, and behaves non-linearly in *S* if non-spherical collapse (ellipsoidal) is considered.

$$B(S,t) = B_0 + B_\gamma S^\gamma = \sqrt{q}\delta_c(t) \left\{ 1 + \beta \left[\frac{q\delta_c^2(t)}{S} \right]^{-\gamma} \right\}$$

Sheth&Tormen99, Mahmood& Rajesh05

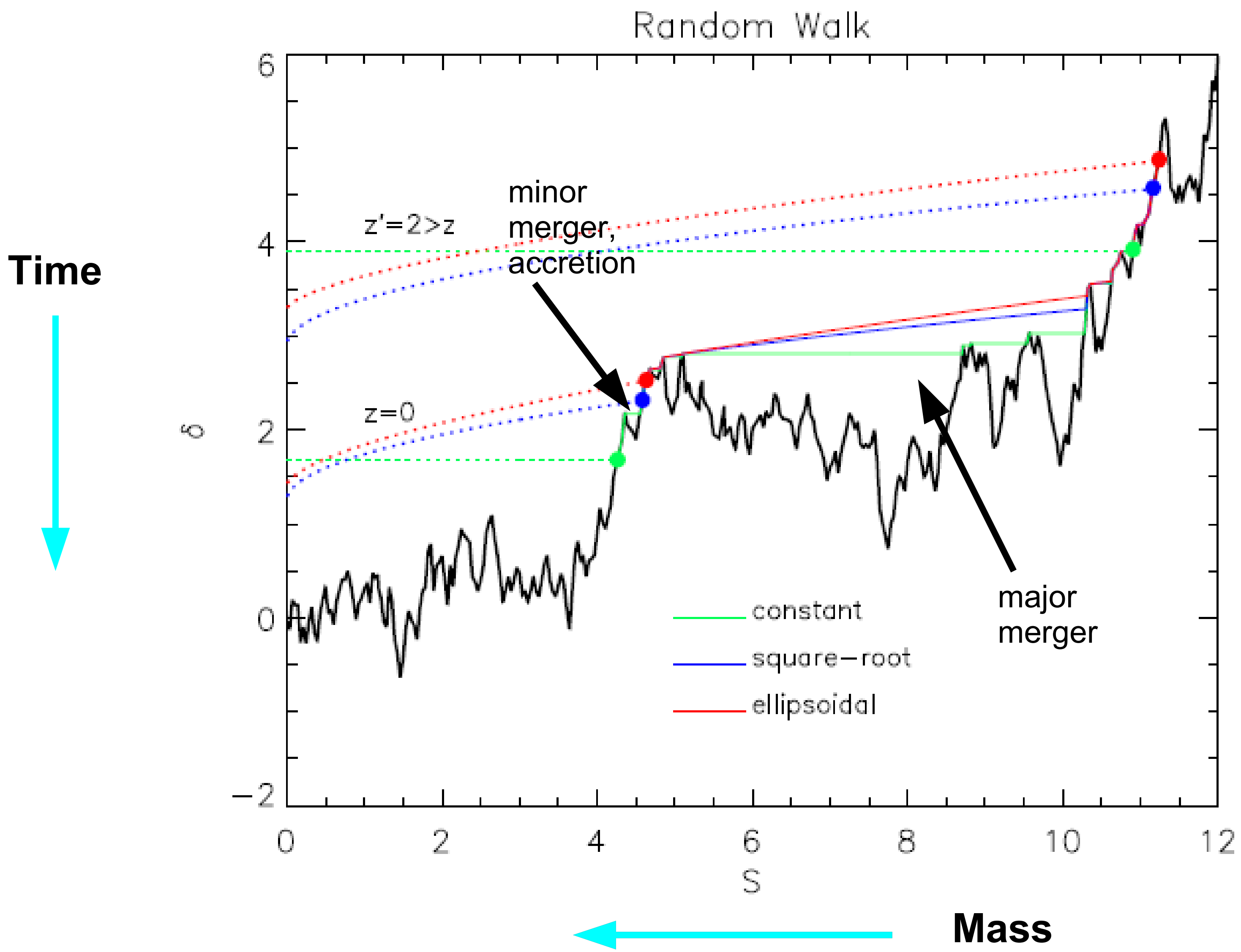
Parameters derived from collapse theory, or from Montecarlo algorithms.

Excursion Set Approach



$$B(S,t) = \sqrt{S}\,(q\nu)^{1/2}\left[1 + \frac{\beta}{(q\nu)^\gamma}\right]$$

Excursion Set Approach



Excursion Set Approach

- To find *f(S)*, it is convenient to formulate the problem in integral terms

Zhang&Hui06

Conservation of trajectories

$$\int_0^S \mathrm{d}S'\ f(S') + \int_{-\infty}^{B(S)} \mathrm{d}\delta\ P(\delta, S) = 1$$

crossed barrier at some S'<S

still below the barrier

In absence of barrier one expects

$$P_0(\delta, S) = \frac{e^{-\delta^2/2S}}{\sqrt{2\pi S}}$$

whereas with a barrier

$$P(\delta, S) = P_0(\delta, S) - \int_0^S \mathrm{d}S'\ f(S')\, P_0[\delta - B(S'), S - S']$$

Fraction of walks now at (δ, S) that crossed the barrier at some S'<S

Excursion Set Approach

One finds the fundamental equation

Lapi+13

$$\mathrm{erfc}\left[\frac{B(S)}{\sqrt{2S}}\right] = \int_0^S \mathrm{d}S'\, f(S')\, \mathrm{erfc}\left[\frac{B(S)-B(S')}{\sqrt{2(S-S')}}\right]$$

Known analytical solution only for linear barrier $B_L(S) = B_0 + B_1 S$

$$f_L(S) = \frac{B_0}{\sqrt{2\pi S^3}}\, e^{-B_L^2(S)/2S}$$

Otherwise, recursive numerical solution (with attendant numerical issues)

$$f(S_i) = \frac{2}{\Delta S_{i-1}}\left\{\mathrm{erfc}\left[\frac{B(S_i)}{\sqrt{2S_i}}\right] - \sum_{j=0}^{i-1} f(S_j)\,\mathrm{erfc}\left[\frac{B(S_i)-B(S_j)}{\sqrt{2(S_i-S_j)}}\right]\frac{\Delta S_{j-1}+\Delta S_j}{2}\right\}$$

Benson+13

Unconditional Mass Function

- Expansion valid at the high/intermediate-mass end

  Differentiate fundamental equation, keep terms to the lowest order in S

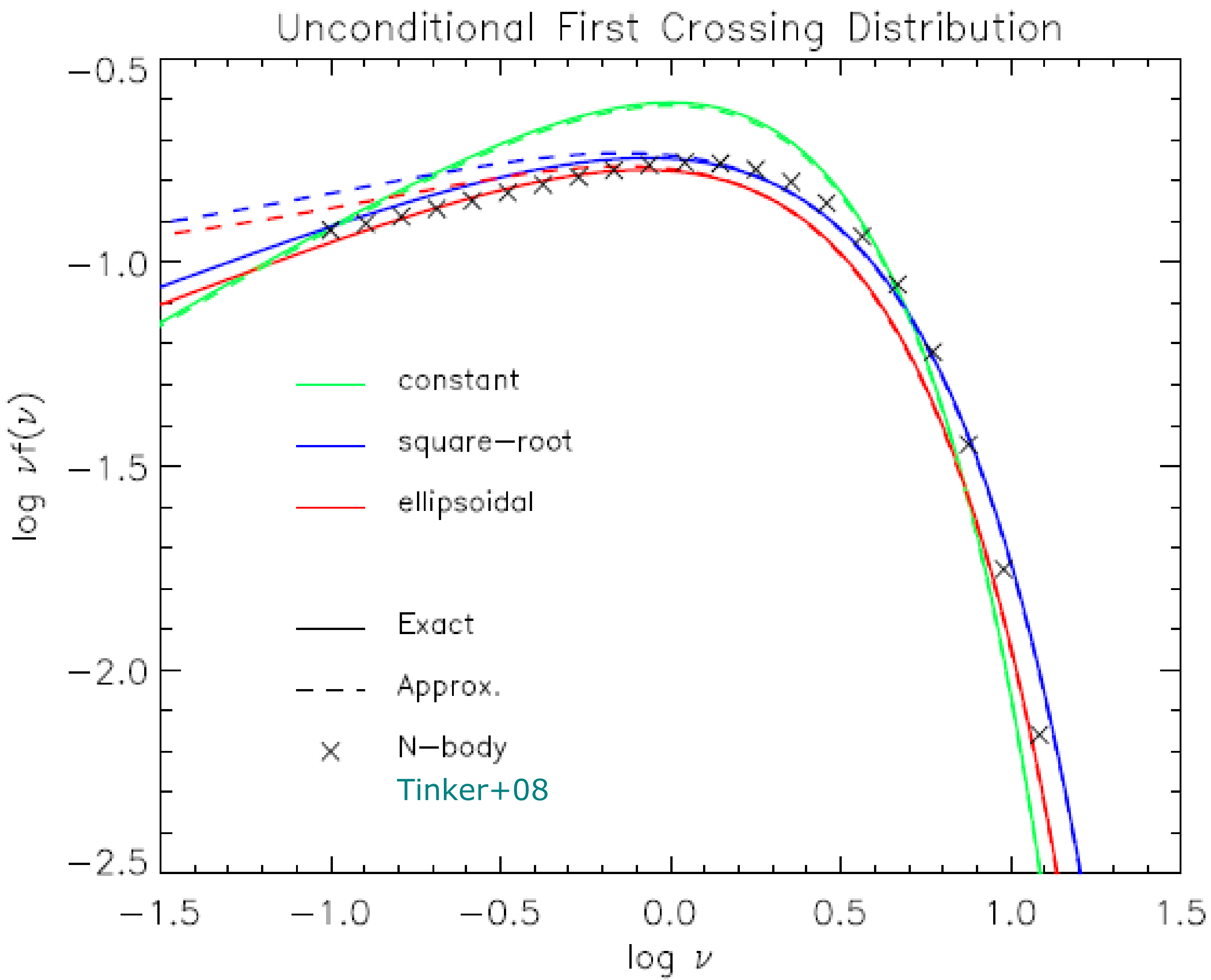
$$\frac{B_0\, e^{-B^2(S)/2S}}{\sqrt{2\pi S^3}}\left[1+(1-2\gamma)\,\frac{B_\gamma}{B_0}\,S^\gamma\right]\simeq f(S)-\frac{B_\gamma\,\gamma\,S^{\gamma-1}}{\sqrt{2\pi}}\int_0^S \mathrm{d}S'\,\frac{f(S')}{\sqrt{S-S'}}\left[1+2\,(\gamma-1)\,\frac{S-S'}{S}\right]$$

  Structure of the above suggests *ansatz* $f(S)\simeq\frac{B_0\, e^{-B^2(S)/2S}}{\sqrt{2\pi S^3}}\left[1+k_\gamma\,\frac{B_\gamma}{B_0}\,S^\gamma\right]$

  Approximating integral by Laplace method yields $k_\gamma = 1-\gamma$ and leads to

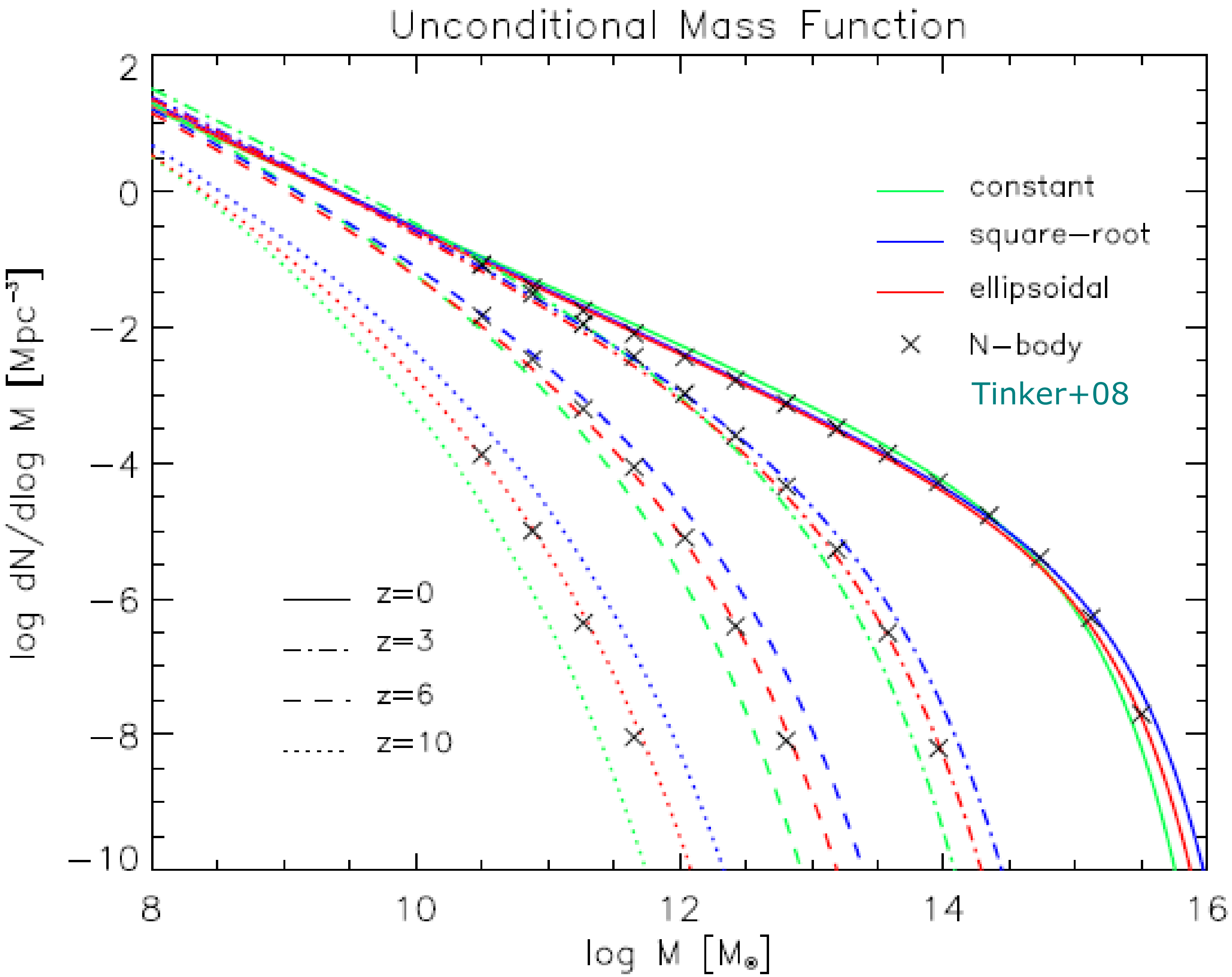
$$f(S)\simeq\frac{B_0\, e^{-B^2(S)/2S}}{\sqrt{2\pi S^3}}\left[1+(1-\gamma)\,\frac{B_\gamma}{B_0}\,S^\gamma\right]$$

Unconditional Mass Function



Unconditional Mass Function

Lapi+13



Conditional Mass Function

- Conditional mass function, distribution of progenitor masses can be derived from a two-barrier problem.

e.g.,Lacey&Cole93, Mahmood&Rajesh05

$$\frac{\mathrm{d}N}{\mathrm{d}M'}(M' \to M, \Delta t) = \frac{M}{M'^2}\,\Delta S\, f(\Delta S)\left|\frac{\mathrm{dlog}\Delta S}{\mathrm{dlog}M'}\right|$$

This is equivalent to find the first crossing distribution for a single barrier

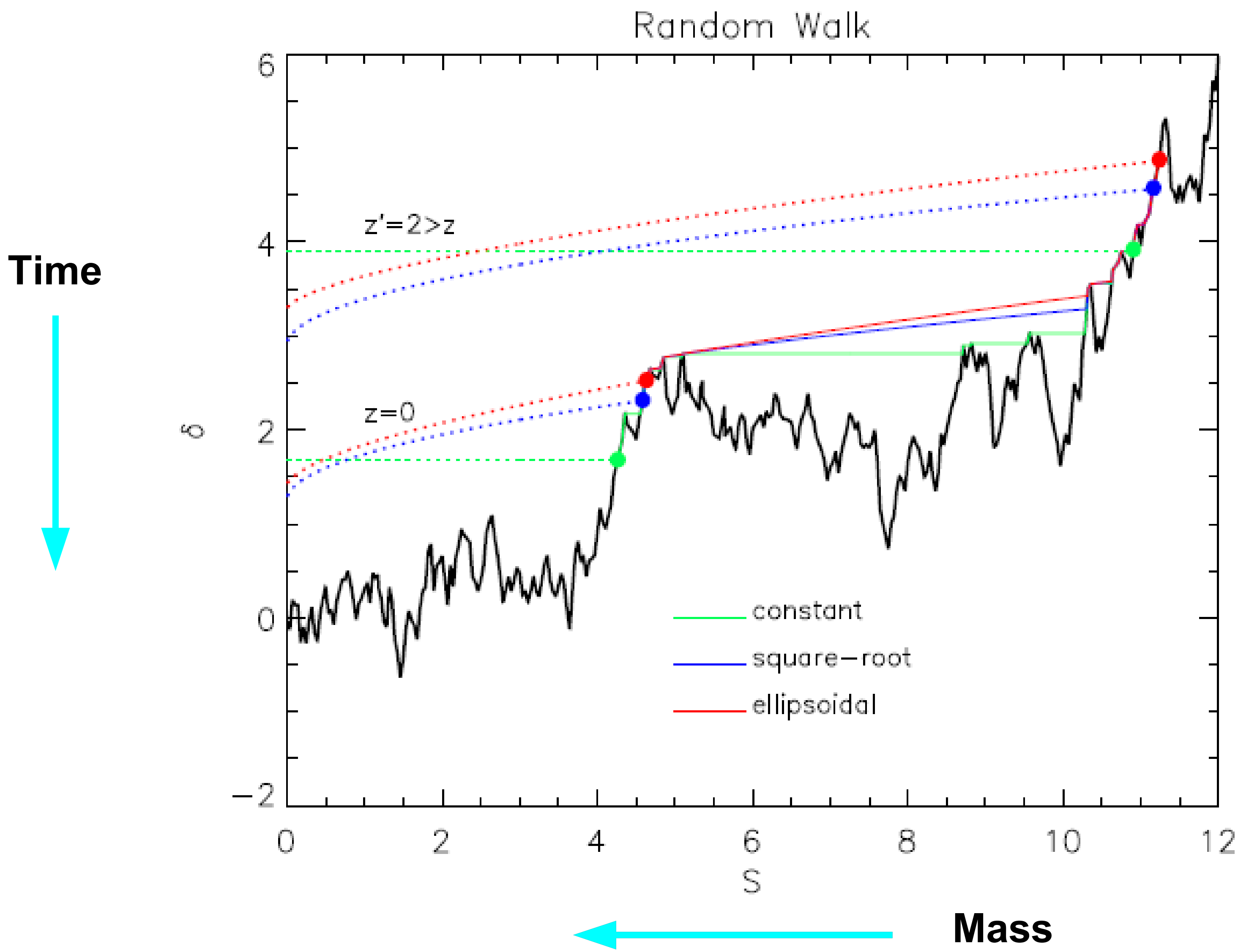
$$B(\Delta S, t, t') = B(S', t') - \bar{B}(S, t)$$

and pseudo-time variable $\Delta S = S' - S$

E.g., if barrier were constant $B(\Delta S) = \delta_c(t') - \delta_c(t) = \Delta\delta_c$

then result would be simply obtain by rescaling unconditional distribution from $\nu \equiv \delta_c^2/S$ to $\nu_c \equiv (\Delta\delta_c)^2/\Delta S$, otherwise result depends on redshift difference.

Conditional Mass Function



Conditional Mass Function

- Expansion of a weakly-varying barrier $B(\Delta S) \simeq C_0 + C_1\,\Delta S + C_2(\Delta S)^2$

  expect first crossing distribution $f(\Delta S) \simeq f_L(\Delta S) + C_2\,\partial_{C_2} f_{|C_2=0}$

  Differentiate fundamental equation

$$-\sqrt{2}\,(\Delta S)^2\,\frac{e^{-(C_0+C_1\Delta S)^2/2\Delta S}}{\sqrt{\pi\Delta S}} = \int_0^{\Delta S} \mathrm{d}\Delta S'\;\partial_{C_2} f_{|C_2=0}(\Delta S')\,\mathrm{erfc}\left(C_1\sqrt{\frac{\Delta S-\Delta S'}{2}}\right) -$$

$$-\sqrt{\frac{2}{\pi}}\int_0^{\Delta S}\mathrm{d}\Delta S'\;f_L(\Delta S')\,\frac{(\Delta S)^2-(\Delta S')^2}{\sqrt{\Delta S-\Delta S'}}\,e^{-C_1^2\,(\Delta S-\Delta S')/2}$$

  Method of Laplace transforms yields

$$\partial_{C_2} f_{|C_2=0}(\Delta S) = -C_0 e^{-(C_0+C_1\Delta S)^2/2\Delta S}\left\{\frac{e^{C_0^2/2\Delta S}}{2}\mathrm{erfc}\left(\frac{C_0}{\sqrt{2\Delta S}}\right) + \frac{C_0+C_1\,\Delta S}{\sqrt{2\pi\Delta S}}\right\}$$

Conditional Mass Function

One finds the approximation

$$f(\Delta S) \simeq \frac{\tilde{C}_0 e^{-[\tilde{C}_0+\tilde{C}_1\Delta S/S]^2/2\Delta S/S}}{\Delta S\sqrt{2\pi\Delta S/S}}\left\{1-\tilde{C}_2\left(\frac{\Delta S}{S}\right)^{3/2}\times\right.$$

$$\left.\times\left[\sqrt{\frac{\pi}{2}}\,e^{\tilde{C}_0^2/2\Delta S/S}\mathrm{erfc}\left(\frac{\tilde{C}_0}{\sqrt{2\Delta S/S}}\right)+\tilde{C}_0\sqrt{\frac{S}{\Delta S}}+\tilde{C}_1\sqrt{\frac{\Delta S}{S}}\right]\right\}$$

in terms of the expansion coefficients

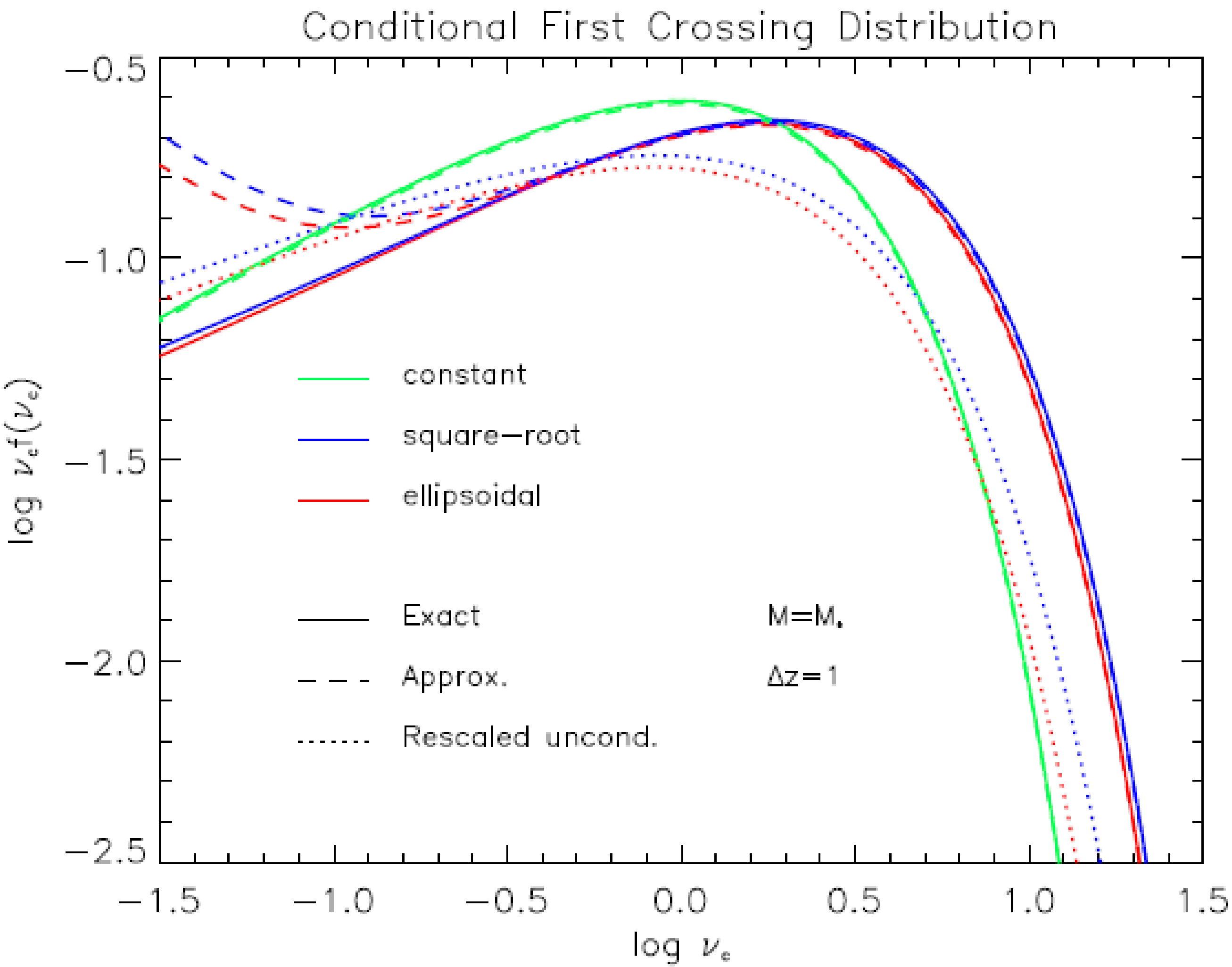
$$\tilde{C}_0 \equiv \frac{C_0}{S^{1/2}} = \left[q\frac{\delta_c^2(t')}{S}\right]^{1/2} - \left[q\frac{\delta_c^2(t)}{S}\right]^{1/2} + \beta\left\{\left[q\frac{\delta_c^2(t')}{S}\right]^{-\gamma+1/2} - \left[q\frac{\delta_c^2(t)}{S}\right]^{-\gamma+1/2}\right\}$$

$$\tilde{C}_1 \equiv C_1\, S^{1/2} = \beta\gamma\left[\frac{q\delta_c^2(t')}{S}\right]^{-\gamma+1/2},$$

$$\tilde{C}_2 \equiv C_2\, S^{3/2} = -\frac{\beta\gamma\,(1-\gamma)}{2}\left[\frac{q\delta_c^2(t')}{S}\right]^{-\gamma+1/2};$$

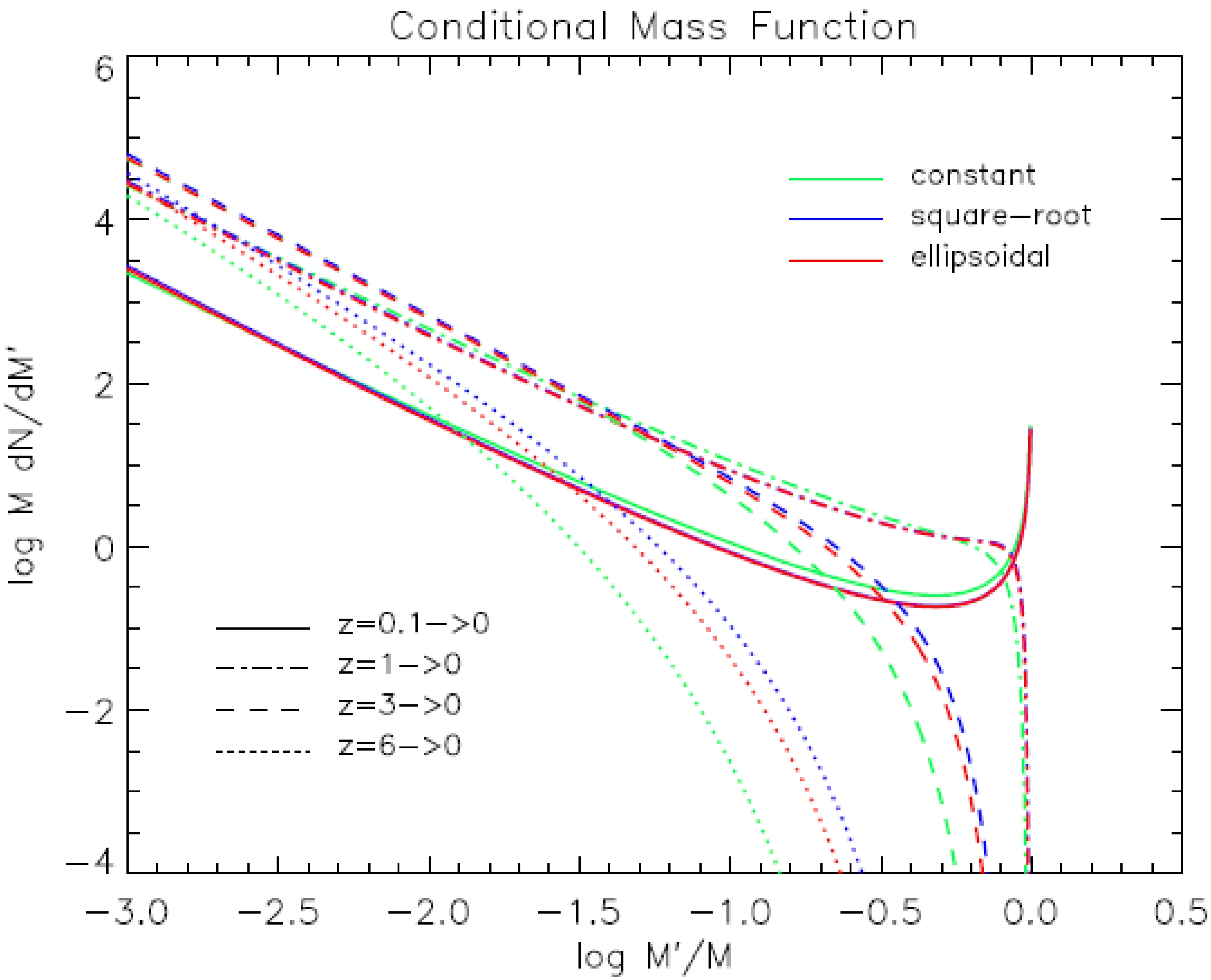
Conditional Mass Function

Lapi+13



Conditional Mass Function

Lapi+13



Merging and Creation Rates

- Rates at which halos merge with others, and at which are created by merging of smaller ones are essential ingredients in galaxy formation models.

One cannot use total derivatives of halo mass function because

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (\partial_t N)_+ - (\partial_t N)_-$$

Cavaliere91
Blain&Longair93
Sasaki94
Haehnelt98

Extracting creation term is a nontrivial task !

Naive solution 1→ just take positive derivative...plainly ambiguous, can be redefined short of an additive constant

Naive solution 2 → require negative term to be scale-invariant...works only for constant barriers, and is not consistent with sims.

What about recurring to Excursion Set Approach ?

Merging and Creation Rates

Starting point is the merging rate → easily derived from small lookback time limit of the conditional mass function Zhang+08, Lapi+13

$$\frac{\mathrm{d}^2 p_{M'\to M}}{\mathrm{d}M'\mathrm{d}t} = \frac{\bar{C}_0\,|\dot{\delta}_c(t)|\,e^{-\bar{C}_1^2\Delta S/2\,S}}{\sqrt{2\pi(\Delta S)^3}}\left|\frac{\mathrm{d}S}{\mathrm{d}M}\right|_{M'}\left\{1-\bar{C}_2\left(\frac{\Delta S}{S}\right)^{3/2}\left[\sqrt{\frac{\pi}{2}}+\bar{C}_1\left(\frac{\Delta S}{S}\right)^{1/2}\right]\right\}$$

in terms of the coefficients

$$\bar{C}_0 \equiv \sqrt{q}\,\left[1-\beta\,(2\gamma-1)\,(q\nu)^{-\gamma}\right]\ ,$$

$$\bar{C}_1 \equiv \beta\,\gamma\,(q\nu)^{-\gamma+1/2}\ ,$$

$$\bar{C}_2 \equiv -\frac{\beta\gamma\,(1-\gamma)}{2}\,(q\nu)^{-\gamma+1/2}$$

Merging and Creation Rates

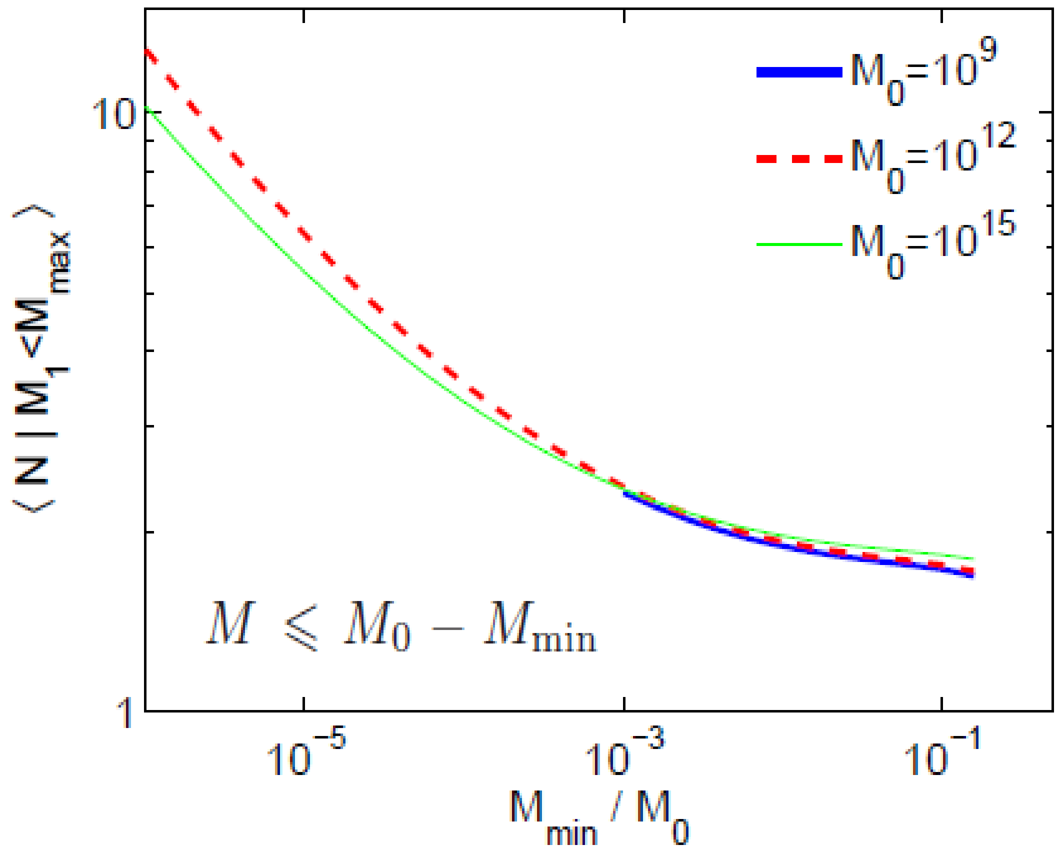
Note that merger rate is not symmetric with respect of M' and M-M' (except for a scale invariant n=0 power spectrum). Two consequences:

Neistein&Dekel08

Assumption of binary mergers is invalid, either because # of mergers >2, or because significant diffuse mass accretion.

A Smoluchowski-like approach to estimate creation rates does not provide consistent results!

Benson+04, 08



Merging and Creation Rates

To compare with N-body outcomes, consider

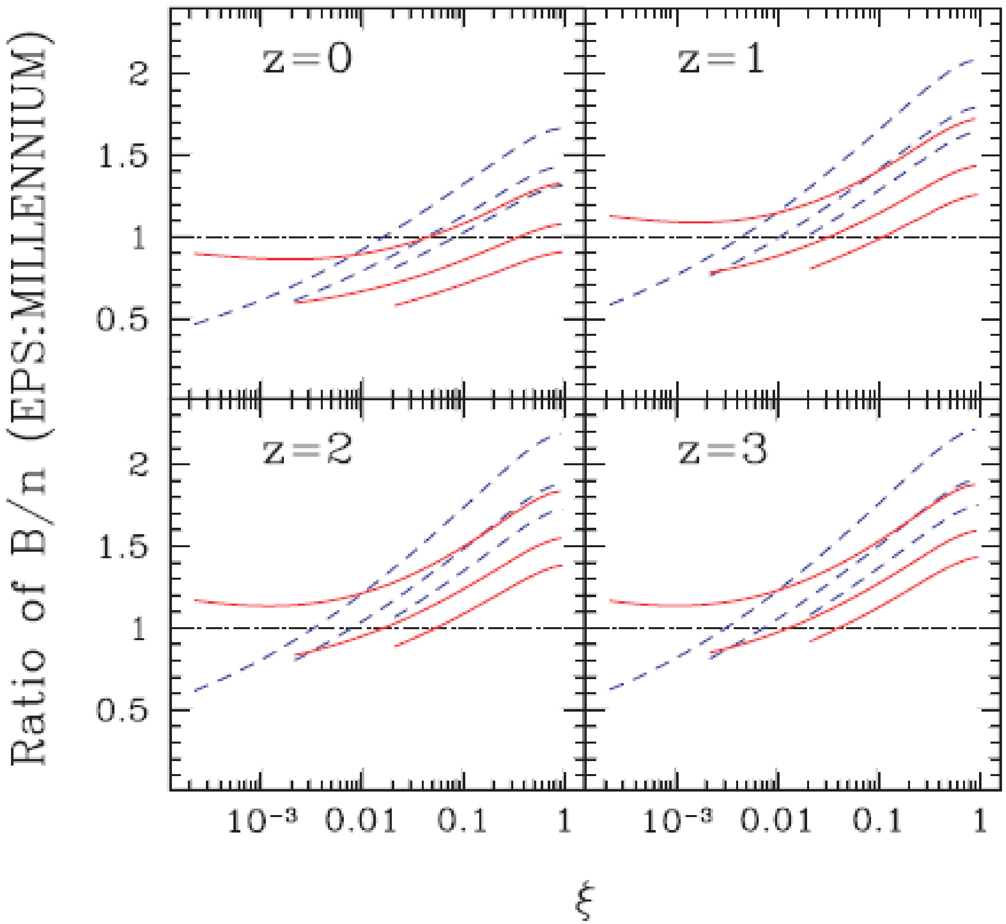
Zhang+08

$$\frac{B(\xi,t)}{N} = \frac{M}{(1+\xi)^2}\,\frac{M}{M_i}\,\frac{\mathrm{d}^2 p_{M_i\to M}}{\mathrm{d}M_i \mathrm{d}t}$$

$$\xi \equiv M_i/(M - M_i)$$

and M_i the less massive progenitor.

Adopting ellipsoidal collapse yields agreement within 20%; likely, *N*-body include also non-Markovian processes.



Merging and Creation Rates

- Creation rates in the excursion set approach are self-consistently defined as

Kitayama&Suto96

$$\partial_t N_+^{\rm crea} = N \int_0^M {\rm d}M' \, \frac{{\rm d}^2 p_{M' \to M}}{{\rm d}M' {\rm d}t}$$

but this expression is found to diverge like *(M-M')*$^{-1/2}$ when *M'* → *M*. This divergence is unphysical, corresponding to the transition of an object into itself.

Standard way of circumventing the problem, instead of “creation rate” compute “formation (or major merger) rate”

Lacey&Cole93

$$\sim M/2$$

$$\partial_t N_+^{\rm form} = N \int_0^{M_f} {\rm d}M' \, \frac{{\rm d}^2 p_{M' \to M}}{{\rm d}M' {\rm d}t}$$

Note this is not fully consistent with Excursion Set $(\partial_t N)_+ - (\partial_t N)_- \neq \frac{{\rm d}N}{{\rm d}t}$

Merging and Creation Rates

Our contribution #1 → Expression of formation rate for a general barrier

Use merging rate and change variable to $x \equiv \sqrt{S/\Delta S}$

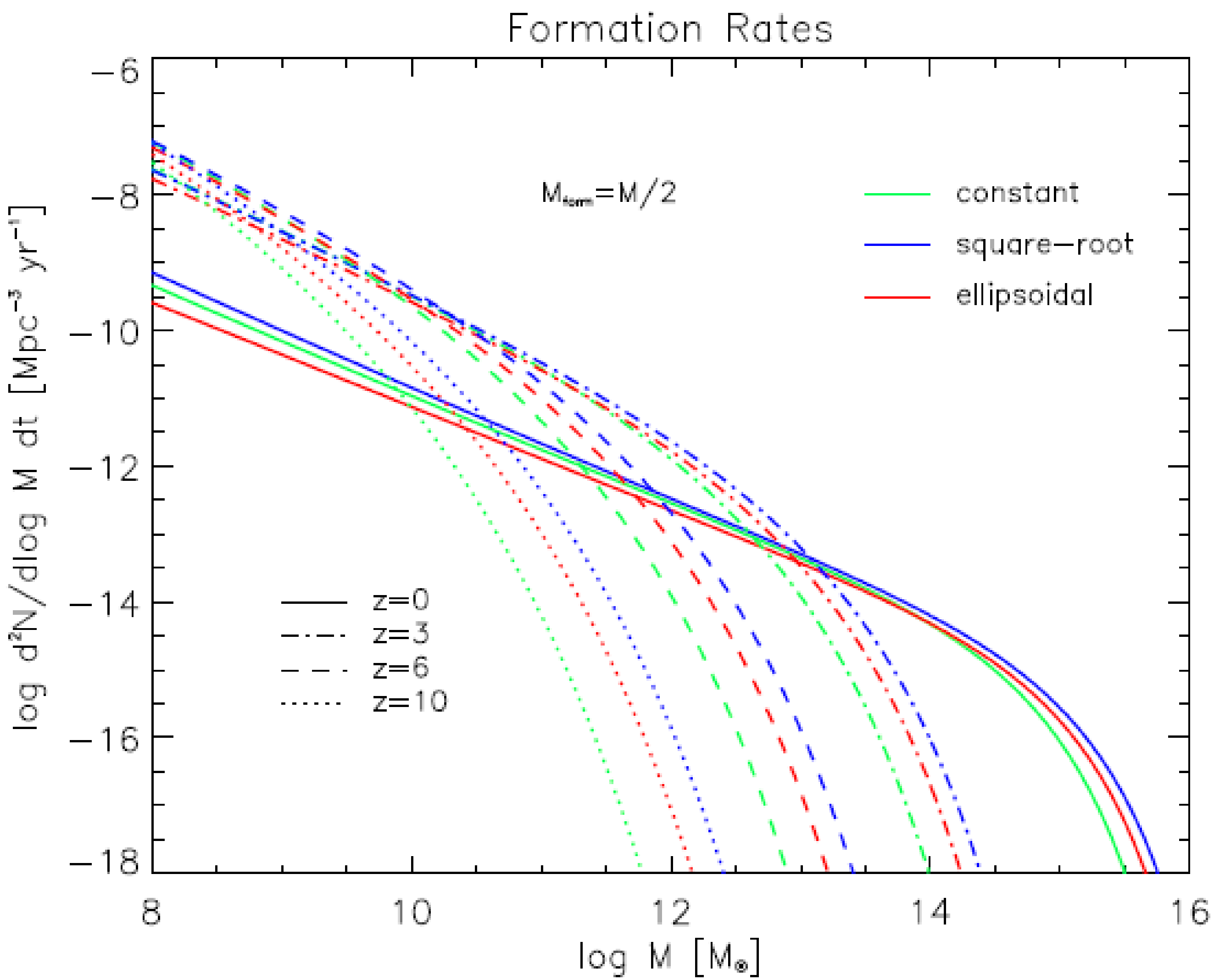
$$\partial_t N_+^{\rm form} = \frac{2\bar{C}_0|\dot{\delta}_c|\, N}{\sqrt{2\pi S}} \int_0^{1/\sqrt{S_{M_f}/S_M-1}} \mathrm{d}x\; e^{-\bar{C}_1^2/2x^2} \left[1 - \sqrt{\frac{\pi}{2}}\, \frac{\bar{C}_2}{x^3} - \frac{\bar{C}_1 \bar{C}_2}{x^4}\right]$$

Integrate term by term and get

$$\begin{aligned} \partial_t N_+^{\rm form} &= \frac{2\bar{C}_0|\dot{\delta}_c|\, N}{\sqrt{2\pi S}} \left[e^{-\bar{C}_1^2\, s_f/2} \left(\frac{1}{s_f^{1/2}} - \sqrt{\frac{\pi}{2}}\, \frac{\bar{C}_2}{\bar{C}_1^2} - \frac{\bar{C}_2}{\bar{C}_1}\, s_f^{1/2}\right) + \right. \\ &\left. - \sqrt{\frac{\pi}{2}} \left(\bar{C}_1 + \frac{\bar{C}_2}{\bar{C}_1^2}\right) \mathrm{erfc}\left(\sqrt{\frac{\bar{C}_1^2\, s_f}{2}}\right)\right], \qquad s_f \equiv S_{M_f}/S_M - 1. \end{aligned}$$

Merging and Creation Rates

Lapi+13



Merging and Creation Rates

Our contribution #2 → Compute creation rates self-consistently with the Excursion Set approach by ζ-regularizing the attendant divergence.

Use merger rates, introduce a regulator Λ >>1 and obtain

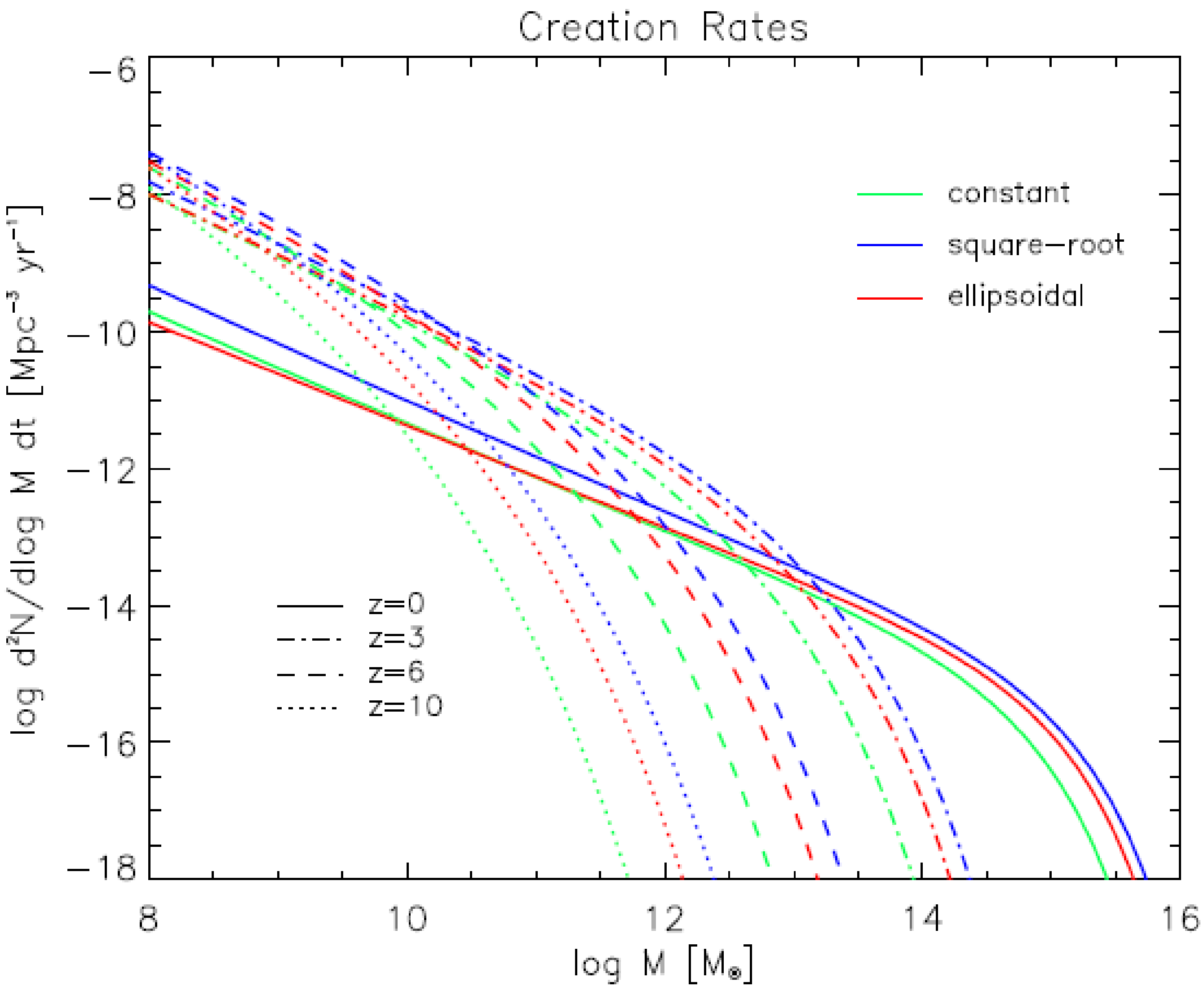
$$\partial_t N_+^{\rm crea} = \frac{2\bar{C}_0|\dot{\delta}_c|\,N}{\sqrt{2\pi S}}\lim_{\Lambda\to\infty}\int_0^{\Lambda\,|{\rm d}\log M/{\rm d}\log S|^{1/2}} {\rm d}x\; e^{-\bar{C}_1^2/2x^2}\left[1-\sqrt{\frac{\pi}{2}}\,\frac{\bar{C}_2}{x^3}-\frac{\bar{C}_1\bar{C}_2}{x^4}\right]$$

ζ-regularize the divergent integral and get

$$\partial_t N_+^{\rm crea} = \frac{\bar{C}_0|\dot{\delta}_c|\,N}{\sqrt{2\pi S}}\left\{\left|\frac{{\rm d}\log M}{{\rm d}\log S}\right|^{1/2}-\sqrt{2\pi}\,\left(\bar{C}_1+2\,\frac{\bar{C}_2}{\bar{C}_1^2}\right)\right\} = \frac{\sqrt{q}\,|\dot{\delta}_c|\,N}{\sqrt{2\pi S}}\,g(\nu)$$

Merging and Creation Rates

Lapi+13



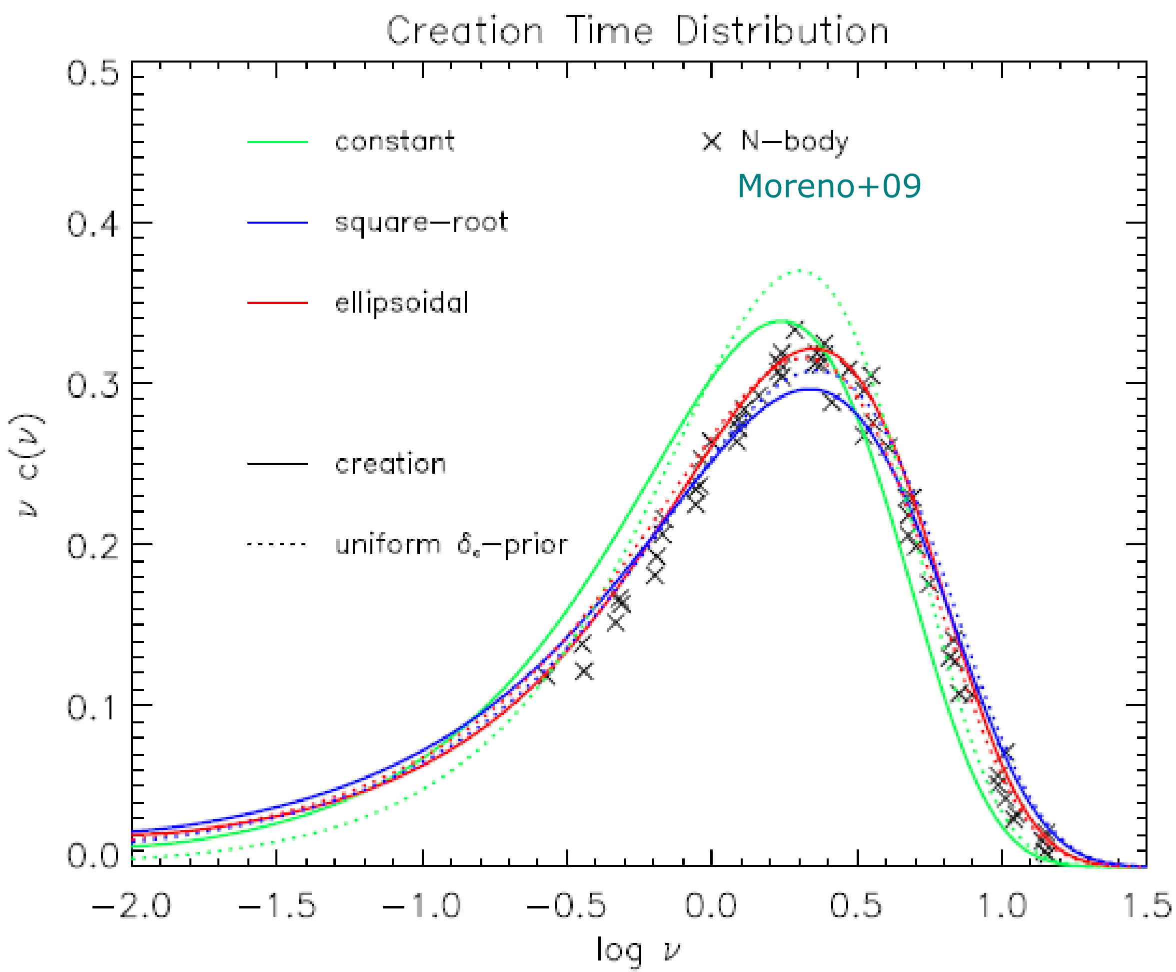
Merging and Creation Rates

Lapi+13

To compare with N-body outcomes, compute time-normalized creation rate, alias creation time distribution.

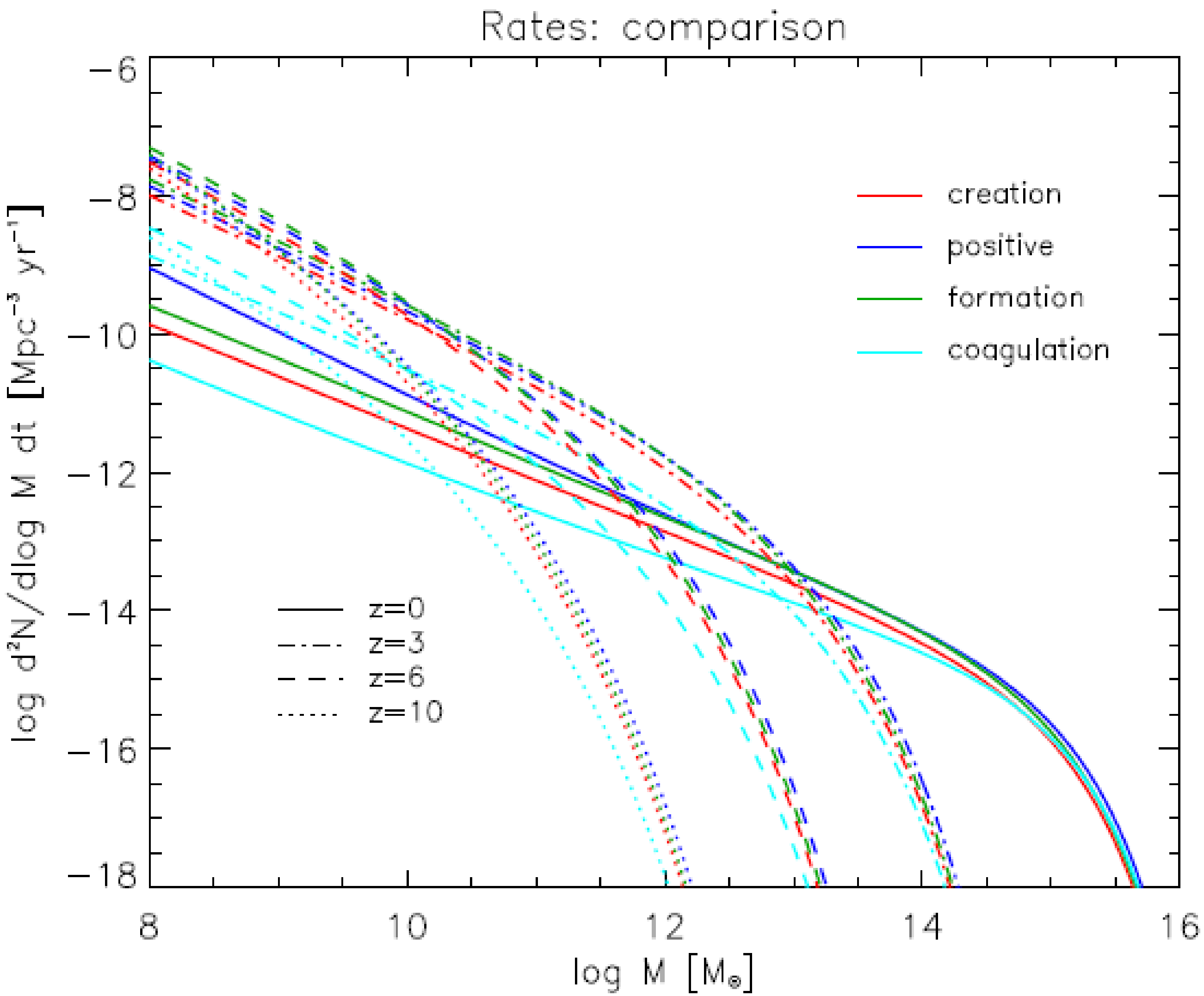
$$c(t|m) = \frac{\partial_t N_+^{\rm crea}}{\int_0^\infty {\rm d}t\, \partial_t N_+^{\rm crea}}$$

for ellipsoidal barrier very good agreement with N-body.



Merging and Creation Rates

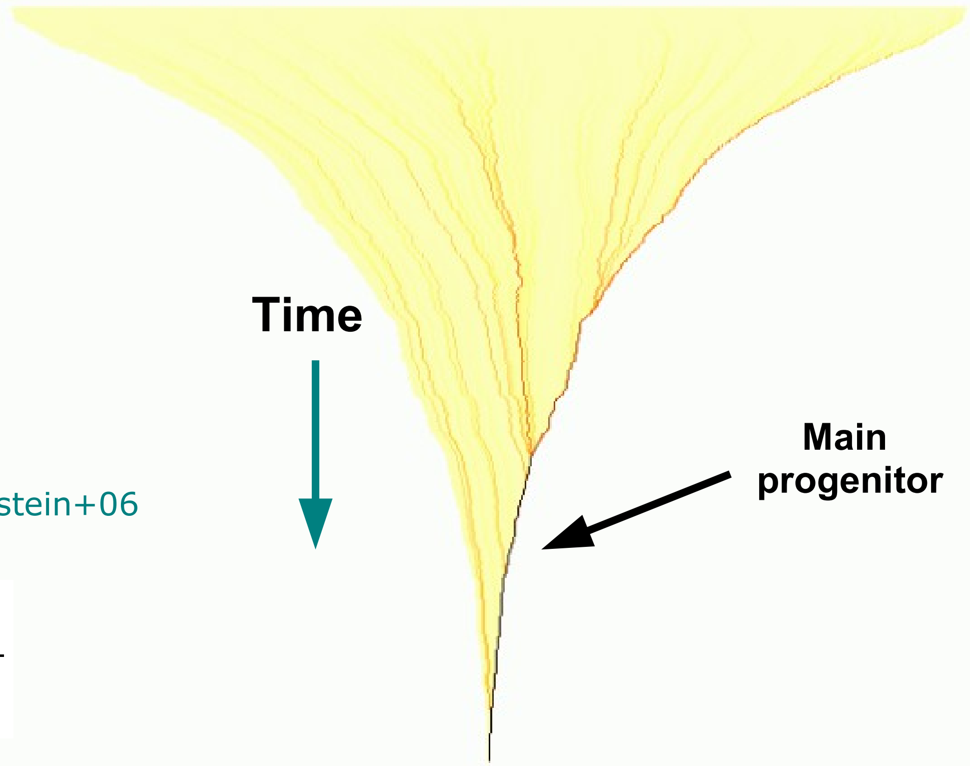
Lapi+13



Mass Growth History

- Given the Excursion Set merger rate, one can build up numerical realization of the merging history of a halo, i.e., "merging tree".

e.g., Menci+12



Can we envisage a faster tool but yielding still very accurate growth histories, at least on the average?

Yes, compute average accretion rate onto mass M at time t

Miller+06, Neistein+06

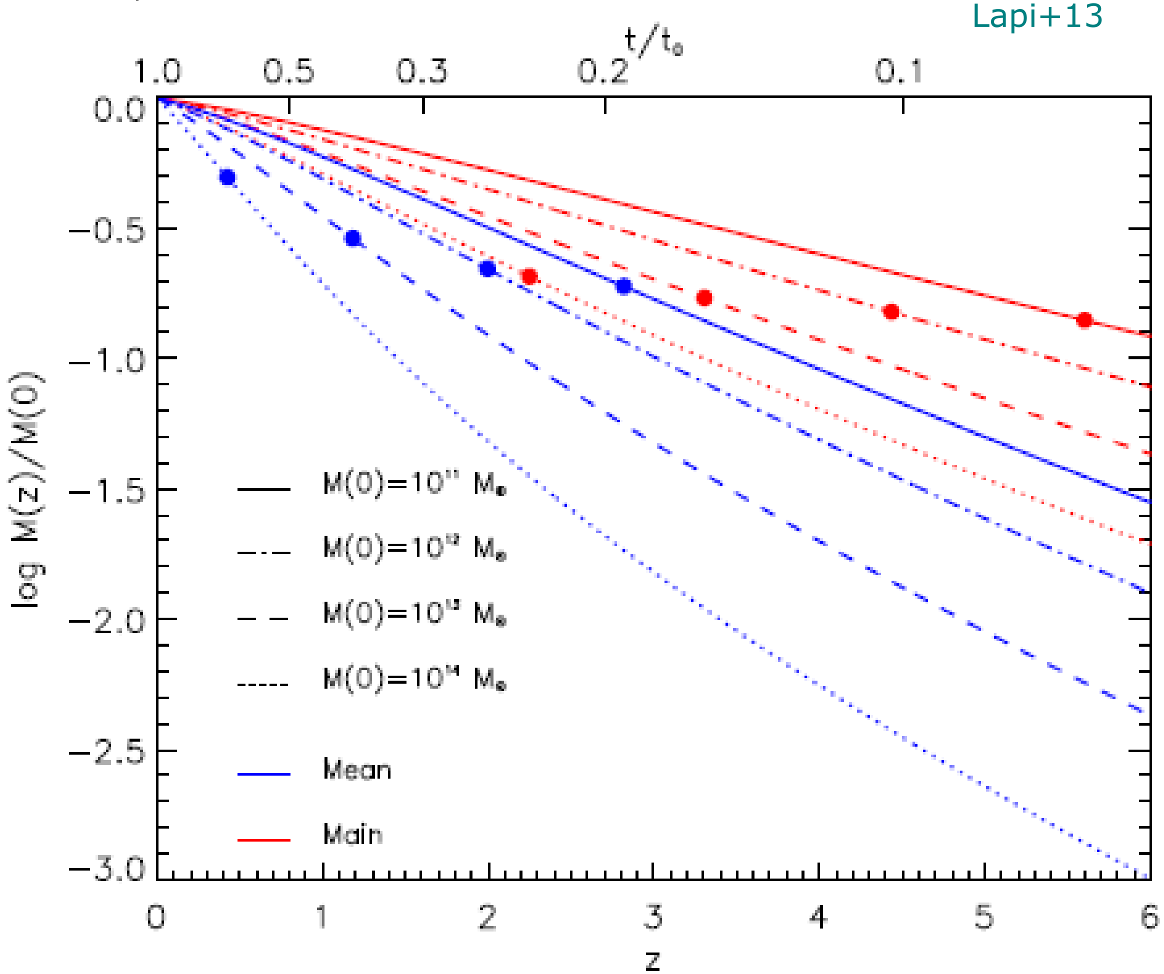
$$\dot{M}(M,t) = \int_{M_{\min}}^{M} \mathrm{d}M'\,(M-M')\,\frac{\mathrm{d}^2 p_{M'\to M}}{\mathrm{d}M'\mathrm{d}t}$$

then integrate backward in time for $M(t)$ with boundary condition $M(t_0) = M_0$

Mass Growth History

As in creation/formation rate issue, one can set

Lapi+13



$M_{min} = 0 \rightarrow$ mean halo

or

$M_{min} = M/2 \rightarrow$ main prog.

We recover two-phase mass growth seen in sims and semianalytic!

→ relevance for galaxy formation models.

e.g., Zhao+03, Fakhouri+10, Wang+11

Mass Growth History

When comparing with N-body outcomes, remember $M_{min} > M_{res}$ to take into account mass resolution, and correct for different cosmology adopted.

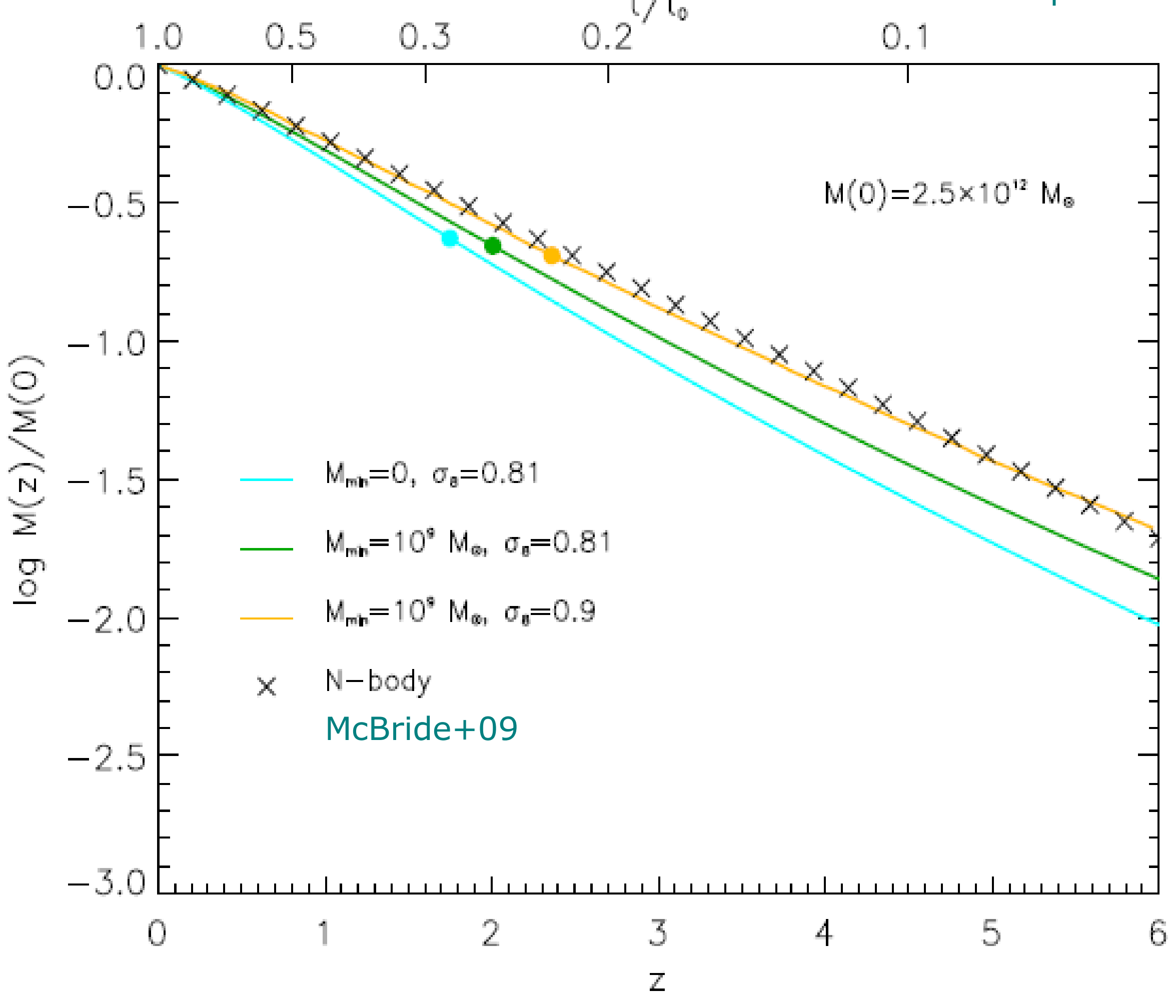
$$M(z)/M(0) = (1+z)^{\beta}\, e^{-\gamma z}$$

$$\beta = 0.1 \qquad \gamma = 0.69$$

for galaxy-sized halos.

Wechsler+02, McBride+09

Lapi+13



Halo bias

- Eulerian halo bias from Excursion Set

Mo&White96, Sheth&Tormen99

$$b(M,z) = 1 + \frac{1}{\delta_{c0}}\left[\frac{N(M,\delta_c \to M_0,\delta_{c0})}{N(M,\delta_c)\,V} - 1\right]$$

Conditions where $\Delta S \sim S >> S_0$ are relevant → conditional tends to unconditional distribution when written in appropriate scaling variable $\nu_c \equiv (\Delta\delta_c)^2/\Delta S$

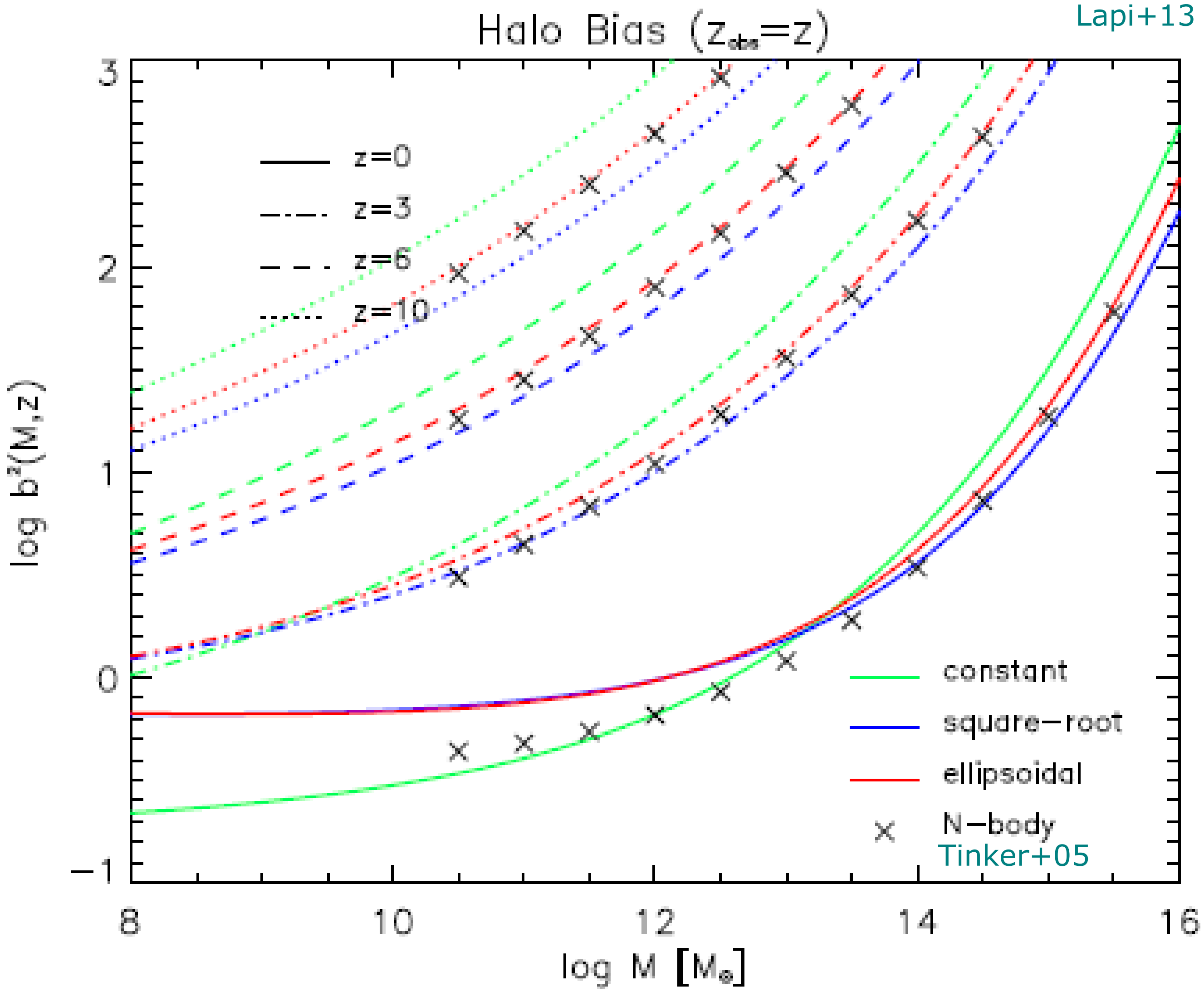
$$B(\Delta S,\delta_c,\delta_{c0}) \,=\, B(S,\delta_c) - B(S_0,\delta_{c0}) \,\simeq\, B(S,\Delta\delta_c)$$

In this limit $\nu_c \simeq \nu(1-2\delta_{c0}/\delta_c)$ $\quad f(\nu_c) \simeq f(\nu) - 2\,(\delta_{c0}/\delta_c)\,\nu\, f'(\nu)$ and one gets

$$b(M,z) = 1 + \frac{1}{\delta_{c0}}\left[\frac{\nu_c\, f(\nu_c)}{\nu\, f(\nu)} - 1\right] \simeq 1 - \frac{2}{\delta_c}\left[1 + \frac{\mathrm{d}\log f}{\mathrm{d}\log \nu}\right]$$

Halo bias

Lapi+13



Warm Dark Matter

Our formulation of the excursion set approach is extremely flexible with respect to changes of the power spectrum → Warm Dark Matter   e.g., Dodelson&Widrow94

This requires to include: Bode+01, Barkana+01

→ cutoff in power spectrum below the free-streaming mass lenght of the particles

→ mass dependent threshold $\delta_c(M,z)$ enforced by particles' residual velocities
Benson+13

Operative procedure, OK above and around the free-streaming mass lenght:

First crossing for ellipsoidal barrier → Connect mass to variance with WDM power spectrum → Introduce $\delta_c(M,z)$ a posteriori

Warm Dark Matter

More details: free streaming...

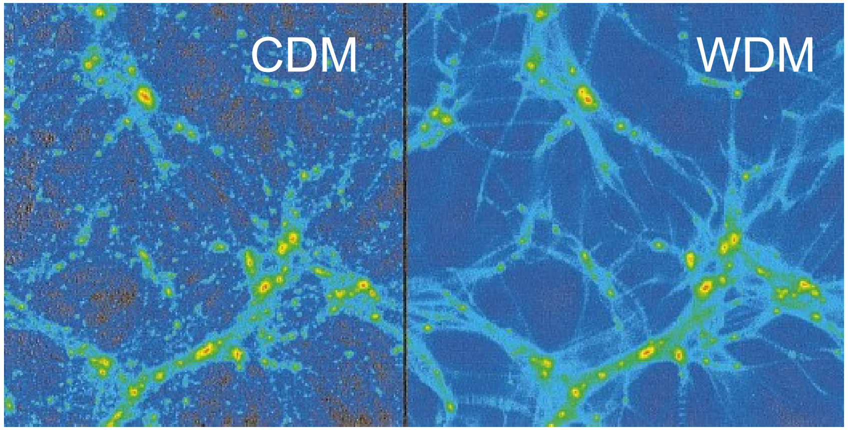
Particles free-stream out of primordial potential well, truncating power on scales below the distance travelled up to ~ radiation-matter equality.

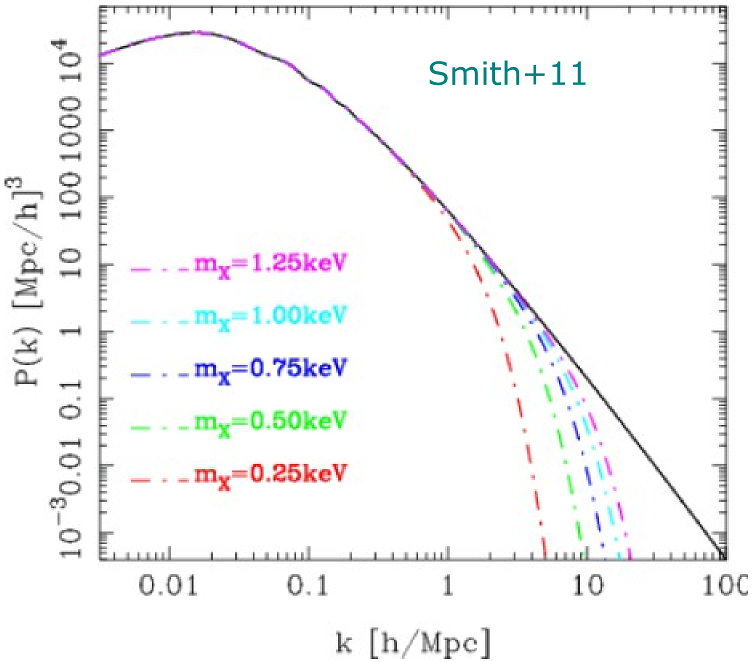
$$R_S \approx 0.31\left(\frac{\Omega_X}{0.3}\right)^{0.15}\left(\frac{h}{0.65}\right)^{1.3}\left(\frac{\text{keV}}{m_X}\right)^{1.15} h^{-1}\ \text{Mpc}$$

$$T(k) \to T(k)\left[1+(\epsilon k \lambda_s)^{2\nu}\right]^{-\eta/\nu}$$

implying a modified primordial power spectrum

Bode+01



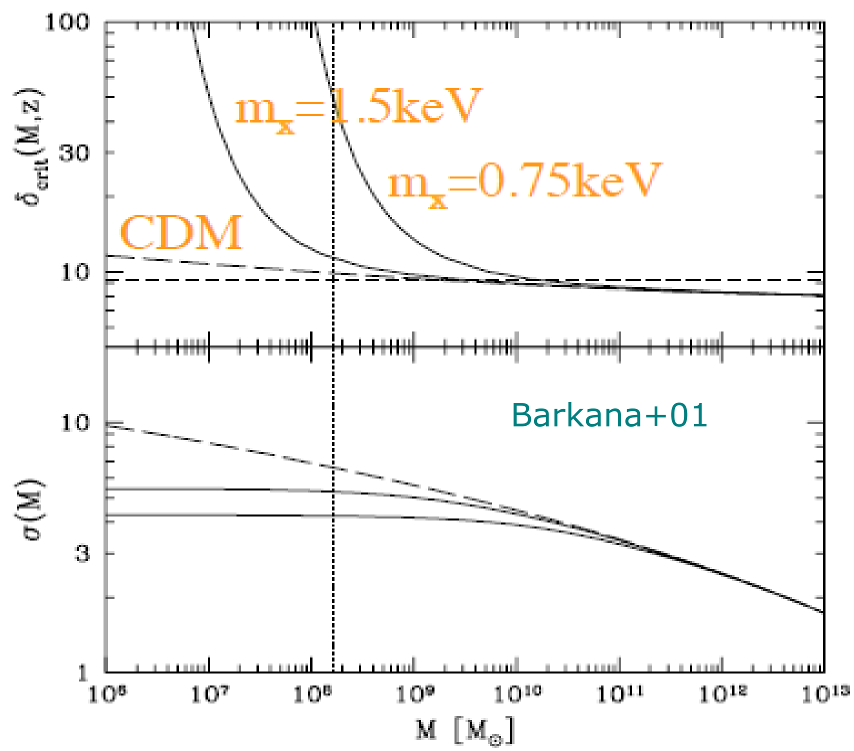


Warm Dark Matter

More details: residual velocities...

Act as an effective pressure, preventing growth of early perturbations below a WDM Jeans mass scale

$$M_J = 3.06 \times 10^8 \left(\frac{1+z_{eq}}{3000}\right)^{1.5} \left(\frac{\Omega_M h_0^2}{0.15}\right)^{1/2} \times \left(\frac{g_X}{1.5}\right)^{-1} \left(\frac{m_X}{1.0\ \mathrm{keV}}\right)^{-4} M_\odot,$$

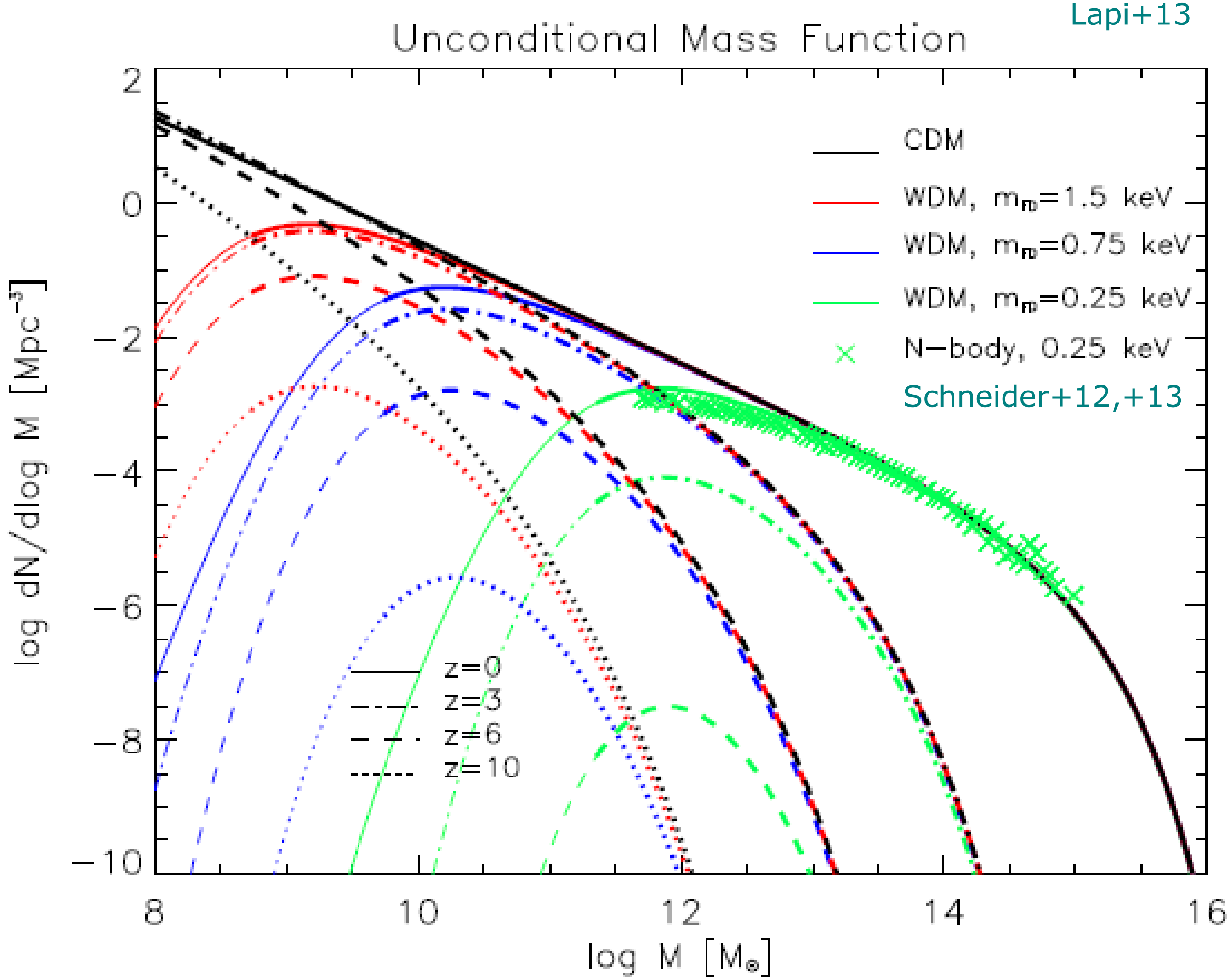


implying a modfied threshold for collapse.

Caveat: in excursion set, ellipsoidal barrier *B(S)* can be trusted only down to M_J

Warm Dark Matter

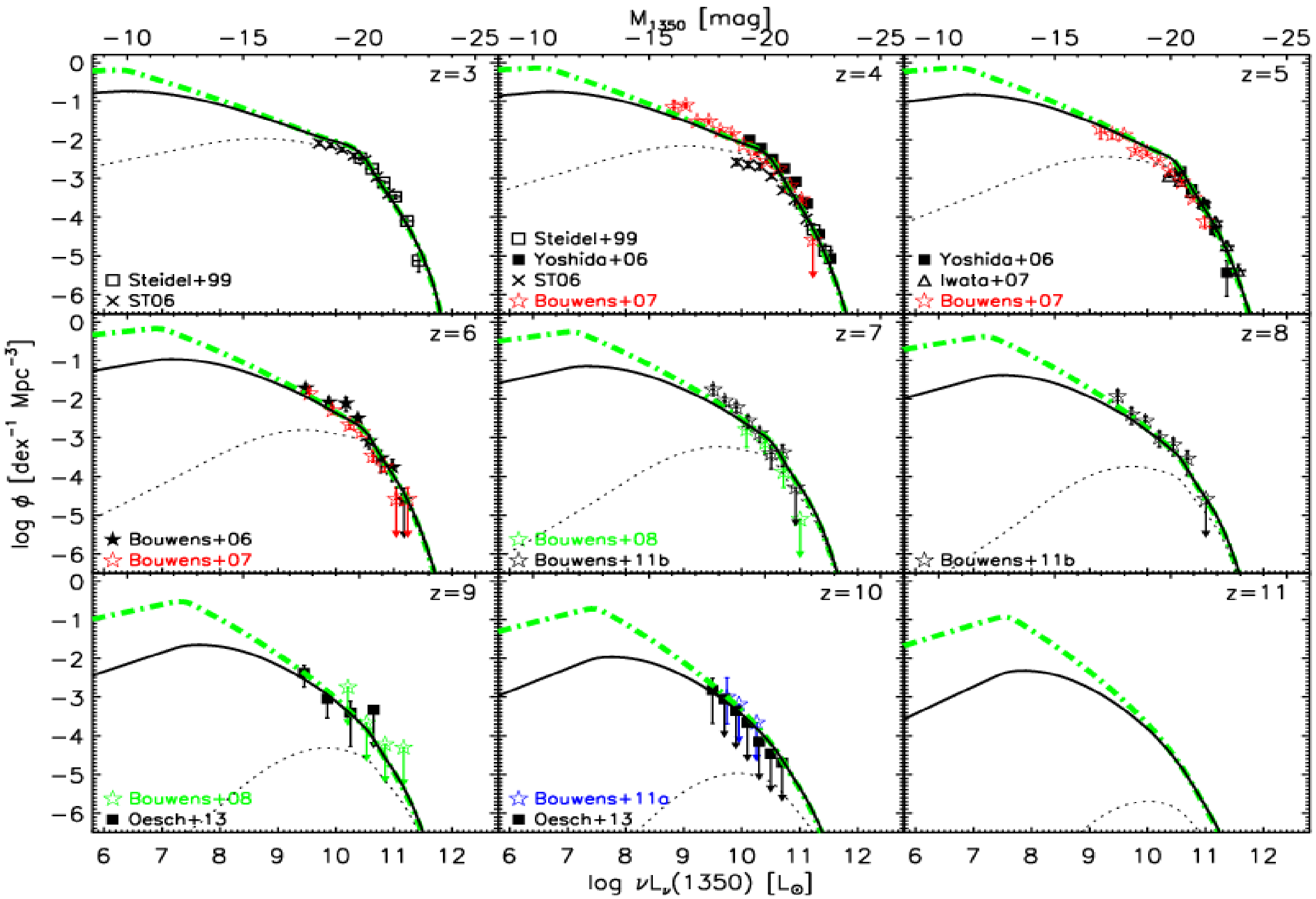
Lapi+13



Warm Dark Matter

Extremely relevant to galaxy formation models in many contexts.

E.g.,Lyman Break Galaxies luminosity functions at high redshift z>6



Cai+13, in prep.

Can the faint end at redshift z~10 constitue a probe of the WDM particle's mass?

Summary

Formation and evolution of DM halos is highly complex, ultimately requires cosmological *N*-body simulations. But some analytic grasp is welcome to:

-) better interpret their outcomes;

-) provide approximated yet flexible analytic representations of the results;

-) compute quantities of extreme relevance in galaxy formation models;

-) develop strategies for future simulation setups;

-) quickly explore effects of varying the cosmological framework.

To these purposes, Excursion Set Approach can provide:

→ accurate approx. of the unconditional/conditional halo mass functions

→ quantities of interest in halo statistics and galaxy formation models, like halo creation rates, average halo growth histories, halo bias

→ quick investigation of Warm Dark Matter cosmogonical scenarios

Correlated Steps?

- What about correlation between the steps? It may result from

  -) more realistic filters in computing mass variance

  -) non-gaussian feature in power spectrum

  -) …

e.g., Maggiore&Riotto10

Paranjape+11,

Musso&Sheth12

To illustrate, consider completely correlated steps. Then δ_c does not "zig-zag", but grows almost monothonically.

$$P(\delta, S) = P_0(\delta, S) - \int_0^S \mathrm{d}S' \; f(S') \, P_0[\delta - B(S'), S - S']$$

One gets fundamental equation

$$\int_0^S \mathrm{d}S' \; f(S') = \frac{1}{2} \,\mathrm{erfc}\left[\frac{B(S)}{\sqrt{2S}}\right]$$

Correlated Steps?

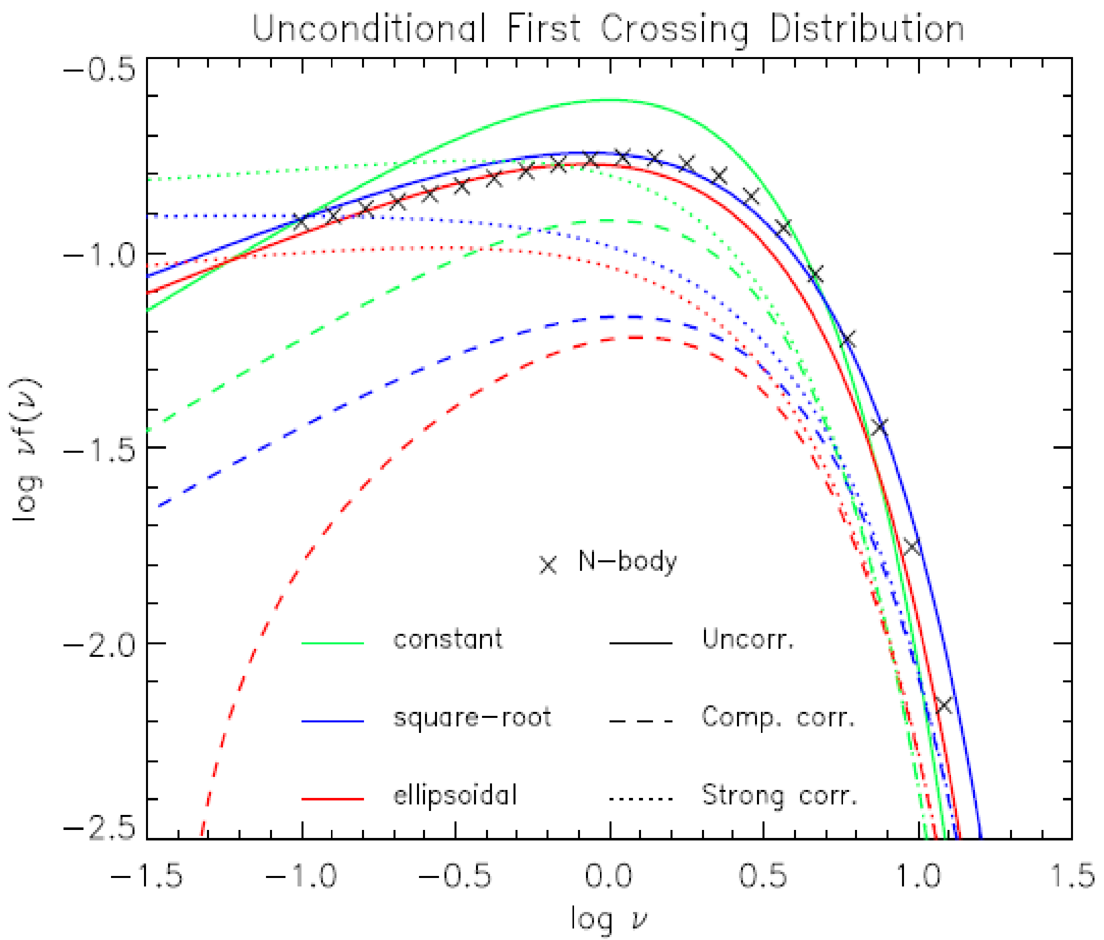
Finally, one gets

$$f(S) = \frac{B_0}{2} \frac{e^{-B^2(S)/2S}}{\sqrt{2\pi S^3}} \left[1 + (1-2\gamma)\frac{B_\gamma}{B_0} S^\gamma\right]$$

which for constant barrier is the old-fashioned P&S result.

Comparison with *N*-body unsatisfactory → improvement may come on considering stochastic barrier (possibly with drift) .

e.g.,Robertson+09, Maggiore&Riotto10, Corasaniti&Achitouv10,11



ζ-regularization

Standard way to regularize divergent integrals, when physical interpretation of it is related to continuous representation of an intrinsically discrete process.

Based on Riemann ζ-function $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

see Hardy49, Birrell&Davies84, Elizalde94

and on its analytic continuation to the whole complex plane through the Dirichlet alternating series and the functional equation

$$\zeta(s) = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \qquad \zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\,\zeta(1-s)$$

which implies $\zeta(0) \simeq \frac{1}{\pi} \frac{\pi s}{2} \frac{1}{(1-s)-1} = -\frac{1}{2}$, and "formally" the weird sum:

$$\zeta(0) = \sum_{n=1}^{\infty} \frac{1}{n^0} = 1 + 1 + 1 + \ldots = -\frac{1}{2}$$

ζ-regularization

Divergent integrals like $\lim_{\Lambda\to\infty}\int_0^\Lambda \mathrm{d}x = \lim_{\Lambda\to\infty}\Lambda = \infty$ can be regularized as

$$\lim_{\Lambda\to\infty}\int_0^\Lambda \mathrm{d}x = \lim_{\Lambda\to\infty}\sum_{n=0}^{\Lambda} 1 = 1 + \lim_{\Lambda\to\infty}\sum_{n=1}^{\Lambda}\frac{1}{n^0} = 1+\zeta(0) = 1-\frac{1}{2} = \frac{1}{2}\,.$$

More involved integral with same diverging behavior may also be regularized by adding and subtracting convenient quantities and using the result above.

In creation rate divergent integral:

$$\begin{aligned}
\lim_{\Lambda\to\infty}\int_0^{\eta\Lambda}\mathrm{d}x\; e^{-k^2/2x^2} &= \lim_{\Lambda\to\infty}\left\{\int_0^{\eta\Lambda}\mathrm{d}x\; e^{-k^2/2x^2} - \eta\Lambda + \eta\int_0^{\Lambda}\mathrm{d}x\right\} = \\
&= \lim_{\Lambda\to\infty}\left\{\left[x\,e^{-k^2/2x^2} + \sqrt{\frac{\pi}{2}}\,k\,\mathrm{erf}\left(\frac{k}{\sqrt{2}x}\right)\right]_0^{\eta\Lambda} - \eta\Lambda\right\} + \frac{\eta}{2} = \\
&= \eta\Lambda - \sqrt{\frac{\pi}{2}}\,k - \eta\Lambda + \frac{\eta}{2} = -\sqrt{\frac{\pi}{2}}\,k + \frac{\eta}{2}\,.
\end{aligned}$$