Strings in cosmological backgrounds

EXAMPLE: **MATTER IN de SITTER (dS) background**

(AdS, BH, BH-dS, BH-AdS)

REGIMES : *classical*, *semiclassical (Q.F.T)*, *quantum (string)*

EFFECTIVE APPROACH

PHASE TRANSITIONS→BOUNDS(T, m, H, Λ, r_g)

A. Bouchareb, M. Ramón Medrano and N.G. Sánchez:

Semiclassical (Quantum Field Theory) and Quantum (String) de Sitter Regimes: New results (Int.J.Mod.Phys. D)

Semiclassical (Quantum Field Theory) and Quantum (String) Anti de Sitter Regimes: New results (Int.J.Mod.Phys. A22, 1395 (2007))

Semiclassical (Quantum Field Theory) and Quantum (String) rotating Black Holes and their evaporation: New results (Int.J.Mod.Phys. A22, 1627 (2007))

M. Ramón Medrano and N. Sánchez, *Phys. Rev D60, 125014*, (1999)

M. Ramón Medrano and N. Sánchez, *Phys. Rev D61, 084030*, (2000)

M. Ramón Medrano and N. Sánchez, *Mod.Phys.Lett A18, 2537* (2003)

N.G. Sánchez, *Int.J.Mod.Phys. A19, 4173*, (2004)

<u>BACKGROUND</u> dS space time metric (static coordinates)

$$ds^2 = -A(r)c^2dt^2 + A^{-1}(r)dr^2 + r^2 d\Omega_{D-2} \quad (1)$$

$$A(r) = 1 - (\frac{r}{L_{c\ell}})^2 \quad , \qquad L_{c\ell} = cH^{-1} \qquad (2)$$

Horizon

$$r_h = L_{c\ell} \qquad (3)$$

H (Hubble constant); $L_{c\ell}$ (classical dS length); cosmological constant $\Lambda \propto H^2/c^2$ (AdS: $H^2 \to -H^2$, $\Lambda \to -\Lambda$; $L_{c\ell} \propto 1/\sqrt{|\Lambda|}$).

(Q.F.T) dS background: a semiclassical (Hawking-Gibbons) temperature T_{sem}

$$T_{sem} = \frac{\hbar c}{2\pi k_B} \, \frac{1}{L_{c\ell}} \qquad (4)$$

$(\forall D)$

QUANTUM STRING and Q.F.T REGIMES:

Quantum string dynamics in dS (AdS)*: string mass spectrum* $m(n, H)$

(H.J. Vega and N. Sánchez *Phys. Lett. B197, 320*, (1987); A.L. Larsen and N. Sánchez *Phys. Rev. D52, 1051*, (1995); H.J. de Vega, A.L. Larsen and N. Sánchez *Phys Rev D51, 6917*, (1995); D58, 26001, (1998))

Microscopic density of mass string states $\rho_s(m, H)$

Quantum String Entropy $S_s(m, H)$

String / Q.F.T **dual** regimes

Phase transitions $\rightarrow$ bounds

QUANTUM STRINGS IN dS

String mass spectrum in dS

$$\left(\frac{m}{m_s}\right)^2 \simeq 4\ n\left[1 - n\left(\frac{m_s}{M_s}\right)^2\right] \tag{5}$$

(closed; $H = 0$, flat space time).

m_s: fundamental string mass; M_s: *characteristic string mass in* dS.

$$m_s = \sqrt{\frac{\hbar}{\alpha' c}}\ ; \qquad M_s = \frac{c}{H\ \alpha'} \tag{6}$$

(α' , string constant; α'^{-1} is a linear mass density; string tension: $\mathcal{T} = c^2/2\pi\alpha'$)

Quantum string dS *length* L_s:

$$L_s = \frac{\hbar}{M_s\ c} \tag{7}$$

String dS *temperature* T_s:

$$T_s = \frac{\hbar c}{2\pi k_B}\ \frac{1}{L_s} \tag{8}$$

String density of levels (level degeneracy, high n)

$$d_n(n) = n^{-a'} \ e^{b \ \sqrt{n}} \tag{9}$$

(a' and b: D, type of strings). Example: closed bosonic strings

$$b = 2\pi\sqrt{\frac{D-2}{6}} \ , \qquad a' = \frac{D+1}{2} \tag{10}$$

String density of states of mass m in dS: $\rho_s(m, H)$

$$\rho_s(m, H) \ d\Big(\frac{m}{m_s}\Big) = d_n(n) \ dn \tag{11}$$

$$\begin{aligned} \rho_s(m, H) \ \ \simeq \ \ & \Big(\frac{m}{\Delta_s M_s}\sqrt{\frac{2}{1-\Delta_s}}\Big) \ \Big(\frac{M_s}{m_s}\sqrt{\frac{1-\Delta_s}{2}}\Big)^{-a} \\ & \times \exp\big(\frac{bM_s}{m_s}\sqrt{\frac{1-\Delta_s}{2}}\big) \end{aligned} \tag{12}$$

$$\Delta_s \equiv \sqrt{1-(\frac{m}{M_s})^2} \tag{13}$$

($H = 0$, flat limit; $a \equiv 2a' - 1 = D$)

ENTROPY of quantum strings in dS

$$\rho_s(m,H) = e^{\frac{S_s(m,H)}{k_B}} \tag{14}$$

(zeroth order) string entropy in flat space time

$$S_s^{(0)}(m) = \frac{1}{2}\ \frac{m\ c^2}{t_s} \tag{15}$$

t_s: fundamental string temperature in flat space-time

$$t_s = \frac{1}{bk_B}\ m_s\ c^2 \tag{16}$$

$$\rho_s(m,H) \simeq \frac{1}{\Delta_s}\sqrt{\frac{1+\Delta_s}{2}}\ \left(\frac{S_s^{(0)}}{k_B}\sqrt{\frac{2}{1+\Delta_s}}\right)^{-a} \times \exp\left(\frac{S_s^{(0)}}{k_B}\sqrt{\frac{2}{1+\Delta_s}}\right) \tag{17}$$

$$\Delta_s \equiv \sqrt{1-4x^2} \tag{18}$$

$$x(m,H) \equiv= \frac{1}{2}\left(\frac{m}{M_s}\right) = \frac{m_s}{bM_s}\frac{S_s^{(0)}}{k_B} \tag{19}$$

String Entropy in dS

$$S_s(m,H) = \hat{S}_s^{(0)}(m,H) - a\ k_B\ \ln\ \left(\frac{\hat{S}_s^{(0)}(m,H)}{k_B}\right) - k_B\ \ln F(m,H) \quad (20)$$

$$\hat{S}_s^{(0)}(m,H) \equiv S_s^{(0)}\sqrt{f(x)} \quad , \quad F \equiv \sqrt{(1-4x^2)f(x)} \quad (21)$$

$$f(x) = \frac{2}{1+\Delta_s} \quad (22)$$

$$S_s(m,H) < S_s(m,0) \equiv S_s^0 \quad (23)$$

(a) $m \ll M_s$, $Hm\alpha'/c \ll 1$: $S_s^{(0)}$, leading term

(b) $m \sim M_s$, $Hm\alpha'/c \sim 1$

$$S_s(m,H)_{m\sim M_s} = k_B \ln \sqrt{\frac{M_s}{(M_s-m)}} - k_B \ln\ 2 + k_B \frac{b}{\sqrt{2}}\ (\frac{M_s}{m_s})\ -\ a k_B \ln\ (\frac{M_s}{m_s})$$

$$S_s(T,H)_{T\sim T_s} = k_B \ln \sqrt{\frac{T_s}{(T_s-T)}} - k_B \ln\ 2 + k_B \frac{b}{\sqrt{2}}\ (\frac{T_s}{t_s})\ -\ a k_B \ln\ (\frac{T_s}{t_s})$$

$$T = \frac{1}{2\pi k_B} mc^2. \qquad (24)$$

String dS **(gravitational) PHASE TRANSITION** : $m = M_s$, $T = T_s$ $(\forall D)$

Critical temperature: T_s $(T_s > t_s)$

$$T_s = \frac{b}{2\pi}\Big(\frac{M_s}{m_s}\Big) t_s \qquad (25)$$

$$T_s = \frac{1}{bk_B}\sqrt{\frac{\hbar c^3}{\alpha'_H}} \quad , \quad \alpha'_H = \frac{\hbar}{c}(\frac{2\pi}{b}\ \frac{H\alpha'}{c})^2 \quad (26)$$

(*Higher effective string tension*: $(\alpha'_H)^{-1} > (\alpha')^{-1}$)

(other examples: thermal self-gravitating gas of point particles, H.J. de Vega and N. Sánchez, *Nucl Phys B625, 409* (2002); B625, 460, (2002); B711, 604, (2005); A. B, M.R.M and N.G.S, strings in flat space time and high spin mode j :$j \to m^2\alpha' c$, $t_{sj} = \sqrt{j/\hbar}t_s, \alpha'_j = \sqrt{\hbar/j}\alpha'$)

(L_s, M_s, T_s), *INTRINSIC* SIZE, MASS and TEMPERATURE of dS **BACKGROUND (String Regime)**

STRING BOUNDS ON THE Q.F.T REGIME

(a) $\underline{dS}$

Partition function (leading order $n = 1$, $\beta_{sem}\, m\, c^2 \gg 1$)

$$\ln Z \;\simeq\; \frac{2V_{D-1}}{(2\pi)^{\frac{D-1}{2}}}\,\frac{1}{(\beta_{sem}\hbar^2)^{\frac{D-1}{2}}} \times \int_{m_0}^{M_s} d(\frac{m}{m_s})\rho_s(m,H)\; m^{\frac{D-1}{2}}\; e^{-(\beta_{sem}mc^2)} \tag{27}$$

$(\beta_{sem} = (k_B T_{sem})^{-1})$

$T_{sem} \to T_s$ (*phase transition*)

$$(\ln Z)_{\sim} V_{D-1}\left(\frac{k_B T_{sem}}{\hbar c}\right)^{D-1}\;\sqrt{1-\frac{T_{sem}}{T_s}} \tag{28}$$

Bounds: $T_{sem} < T_s$, $H < c/L_s(H,\alpha')$ ($H < c^2 m_s/\hbar$; $H_s = c/l_s$)

BACKGROUND → STRING

(b) $\underline{BH - dS}$

$$A(r) = 1 - \frac{r_g}{r} - (\frac{r}{L_{c\ell}})^2 \qquad (29)$$

(r_g: Schwarzschild gravitational radius) Hawking Temperature

$$T_{sem\ bhdS} = \frac{\hbar}{2\pi k_B c} \ \ \mathcal{K}_{bhdS} \qquad (30)$$

($\mathcal{K}_{bhdS}$: surface gravity)

$$T_{sem\ bhdS} = \frac{\hbar c}{2\pi k_B} \ \ \frac{1}{L_{bhdS}} \qquad (31)$$

$$L_{bhdS} = 2 \ r_g \Big(1 - 2 \ \frac{r_g^2}{L_{c\ell}^2}\Big)^{-1} \qquad (32)$$

Quantum emission

$T_{sem\ bhdS} \ll T_s$: Q.F.T Hawking emission

$T_{sem\ bhdS} \sim T_s$

$$\sigma_{string} \ \ (T \sim T_s) \ \ \sim \ \ V_{D-1} \ \Gamma_A \ \left(\frac{k_B T_{sem\ bhdS}}{\hbar c}\right)^{D-1} \times \sqrt{1 \ - \ \frac{T_{sem\ bhdS}}{T_s}} \qquad (33)$$

STRING / Q.F.T: DUAL REGIMES

$$(dS)_{sem} = (L_{c\ell}, T_{sem}, \rho_{sem}, S_{sem}) \longrightarrow (dS)_s = (L_s, T_s, \rho_s, S_s) \quad (34)$$

SEMICLASSICAL dS ENTROPY

Zeroth order (dS Bekenstein-Gibbons-Hawking)

$$S^{(0)}_{sem}(H) = \frac{1}{2}\ \frac{M_{cl}\ c^2}{T_{sem}} \quad (35)$$

dS ENTROPY

$$S_{sem}\ (H) = \hat{S}^{(0)}_{sem}\ (H) - a\ k_B\ \ln\ (\frac{\hat{S}^{(0)}_{sem}\ (H)}{k_B}) - k_B \ln\ F(H) \quad (36)$$

$$\hat{S}^{(0)}_{sem}(H) \equiv S^{(0)}_{sem}(H)\sqrt{f(X)}; F(H) \equiv \sqrt{(1-4X^2)f(X)} \quad (37)$$

$$f(X) = \frac{2}{1+\Delta}, \Delta \equiv \sqrt{1-4X^2} = \sqrt{1-(\frac{\pi k_B}{S^{(0)}_{sem}(H)})^2} \quad (38)$$

$$\ell_{pl} = \sqrt{\frac{\hbar\ G}{c^3}} \qquad m_{pl} = \sqrt{\frac{\hbar\ c}{G}} \qquad (D=4) \quad (39)$$

$$2X(H) \equiv \frac{\pi k_B}{S^{(0)}_{sem}(H)} = (\frac{m_{Pl}}{M_{cl}})^2 \qquad (D=4) \quad (40)$$

$$M_{cl} = \frac{c^2}{G}\ L_{c\ell} \qquad T_{sem} = \frac{c^2}{2\pi k_B}\ \frac{m^2_{pl}}{M_{c\ell}} \qquad (D=4) \quad (41)$$

low $H \ll c/\ell_{Pl}$ (low curvature regime; $M_{cl} \gg m_{Pl}$, $L_{c\ell} \gg \ell_{Pl}$)

$$S_{sem}(H) = S^{(0)}_{sem}(H) - ak_B\ \ln(\frac{S^{(0)}_{sem}(H)}{k_B}) \quad (42)$$

high H: $H \sim c/\ell_{Pl}$ ($M_{cl} \sim m_{Pl}$, $L_{cl} \sim l_{Pl}$: quantum gravity domain)

$$S_{sem}(H)_{\Delta \sim 0} = k_B\ \ln \Delta\ +\ O(1) \qquad (43)$$

$$\Delta\ =\ \sqrt{1 - (\frac{m_{Pl}}{M_{cl}})^4} \qquad (44)$$

phase transition $M_{cl} \to m_{Pl}$ (examples: thermal self-gravitating gas of point particles; Kerr BH entropy, $J \to M^2G/c$, $M \to \sqrt{J/\hbar}\ \ m_{Pl}$ (extremal transition): A. B, M. R. M and N. G. S)$(\forall D)$

AdS BACKGROUND

String density of mass levels in AdS

$$\frac{m}{m_s} \simeq 2\ n\Big(\frac{1}{n} + \big(\frac{m_s}{M_s}\big)^2\Big)^{\frac{1}{2}} \tag{45}$$

$$\rho_s(m,|\Lambda|) \simeq \frac{1}{\Delta_s}\sqrt{\frac{1+\Delta_s}{2}}\ \big(\frac{S_s^{(0)}}{k_B}\sqrt{\frac{2}{1+\Delta_s}}\big)^{-a} \exp\big(\frac{S_s^{(0)}}{k_B}\sqrt{\frac{2}{1+\Delta_s}}\big) \tag{46}$$

$$x(m,|\Lambda|) \equiv \frac{1}{2}(\frac{m}{M_s}) = \frac{m_s}{bM_s}\frac{S_s^{(0)}}{k_B} \tag{47}$$

$$\Delta_s \equiv \sqrt{1+4x^2} \tag{48}$$

no singular behavior

String entropy in AdS

$$S_s(m,|\Lambda|) = \hat{S}_s^{(0)}(m,|\Lambda|) - ak_B \ln(\frac{\hat{S}_s^{(0)}(m,|\Lambda|)}{k_B}) - k_B \ln F(m,|\Lambda|) \quad (49)$$

$$\hat{S}_s^{(0)}(m,|\Lambda|) \equiv S_s^{(0)} \sqrt{f(x)} \quad , \quad f(x) = \frac{2}{1+\Delta_s} \quad (50)$$

$$F(m,|\Lambda|) \equiv \sqrt{(1+4x^2)f(x)} \quad (51)$$

$$S_s(m,|\Lambda|) > S_s(m,0) \equiv S_s^{(0)}(m) \quad (52)$$

$m \sim M_s$ $(Hm\alpha'/c \sim 1)$, $m \gg M_s$ $(Hm\alpha'/c \gg 1)$

$$S_s(T \sim T_s) = k_B \left(\frac{T_s}{t_s}\right) - ak_B \ln\left(\frac{T_s}{t_s}\right) \quad (53)$$

No critical temperature. No phase transition at T_s

$$S_s(T \gg T_s) = k_B \left(\frac{1}{t_s}\sqrt{\frac{T\ T_s}{2}}\right) - \left(\frac{1+a}{2}\right) k_B \ln\left(\frac{T}{t_s}\right) - \left(\frac{1-a}{2}\right) k_B \ln \left(\frac{T_s}{t_s}\right) \quad (54)$$

Semiclassical (Q.F.T) AdS Entropy

$$S_{sem}\ (|\Lambda|) \;=\; \widehat{S}^{(0)}_{sem}\ (|\Lambda|) \tag{55}$$

$$-a\ k_B\ \ln\ (\frac{\widehat{S}^{(0)}_{sem}\ (|\Lambda|)}{k_B}) - k_B \ln\ F(|\Lambda|)$$

$$\widehat{S}^{(0)}_{sem}\ (|\Lambda|) \equiv S^{(0)}_{sem}\ (|\Lambda|)\sqrt{f(X)}\ ,\ \ f(X) = \frac{2}{1+\Delta} \tag{56}$$

$$F(|\Lambda|) \equiv \sqrt{(1+4X^2)f(X)} \tag{57}$$

$$2X(|\Lambda|) \equiv \frac{\pi k_B}{S^{(0)}_{sem}(|\Lambda|)} = (\frac{m_{Pl}}{M_{cl}})^2 \tag{58}$$

Low curvature , $M_{cl} \gg m_{Pl}$

$$S_{sem}(|\Lambda|) = S^{(0)}_{sem}(|\Lambda|) - a k_B\ \ln(\frac{S^{(0)}_{sem}(|\Lambda|)}{k_B}) \tag{59}$$

High curvature $M_{c\ell} \sim m_{Pl}$

$$\Delta = \sqrt{1 + (\frac{m_{Pl}}{M_{cl}})^4} \qquad (60)$$

$M_{c\ell} \to m_{Pl}$ $(T \to t_{Pl})$ $(S^{(0)}_{sem}(|\Lambda|)(M_{c\ell} = m_{Pl}) = \pi k_B)$

$$S_{sem}(|\Lambda|) = \sqrt{\frac{2}{1+\sqrt{2}}}\ \pi k_B - a k_B\ \ln(\sqrt{\frac{2}{1+\sqrt{2}}}\ \pi) - k_B\ \ln(\frac{2}{\sqrt{1+\sqrt{2}}}) \qquad (61)$$

NO PHASE TRANSITION

$M_{cl} << m_{Pl}$ ($H \gg c/l_{Pl}$; $S^{(0)}_{sem}(H)(X \gg 1) = \pi k_B/2X \ll 1)$

$$S_{sem}(H)(X >> 1) = a k_B \ln(\frac{2}{\pi} X^{3/2}) \qquad (62)$$

$$-k_B \ln\left(2X^{1/2}\right) + \frac{\pi k_B}{2X^{3/2}} \qquad (63)$$

BH-dS bounds

$T_{sembhdS} < T_s$

$$H \; [1 - 2r_g^2 \left(\frac{H}{c}\right)^2] < \frac{2r_g c}{\ell_s^2} \tag{64}$$

$(l_s = \sqrt{\frac{\hbar\alpha'}{c}} = \frac{\hbar}{m_s c})$

$T_{sembhdS} \rightarrow T_s$

$$r_g = \frac{1}{2} \; \frac{L_{c\ell}^3}{\ell_s^2} \; [-1 \; + \; \sqrt{1 + 2\left(\frac{\ell_s}{L_{c\ell}}\right)^4} \;] \tag{65}$$

$L_{c\ell} \gg \ell_s$

$$2r_g \simeq \frac{H}{c}\ell_s^2 = L_s \tag{66}$$

$L_{c\ell} = \ell_s$

$$2r_g = 0.73 \; \ell_s \tag{67}$$

BH-AdS *bounds*

$T_{sembhAdS} = T_s$

$$r_{g\pm} \ = \ \frac{1}{2}\ \frac{L_{c\ell}^3}{\ell_s^2}\ [\ 1\ \pm\ \sqrt{1-2\left(\frac{\ell_s}{L_{c\ell}}\right)^4}\ ] \quad (68)$$

(physical roots) $(L_s = (\ell_s)^2/L_{c\ell})$

$$L_{c\ell}\ \geq 2^{1/4}\ \ell_s\ \equiv L_{c\ell\ min}\ =\ 1.189\ \ell_s \quad (69)$$

$$L_s\ \leq\ 2^{-1/4}\ell_s\ \equiv\ L_{s\ max}\ =\ 0.841\ \ell_s. \quad (70)$$

$L_{c\ell} \gg L_{c\ell\ min}\ (L_s \ll L_{smax})$

$$r_{g+}\ \simeq \frac{L_{c\ell}^2}{L_s} = \frac{c^4}{\hbar\alpha' H^3} \quad , \quad r_{g-} \simeq \frac{1}{2}\ L_s = \frac{\hbar\alpha' H}{2c^2} \quad (71)$$

$L_{c\ell}\ =\ L_{c\ell\ min}\ (L_s\ =\ L_{s\ max})$

$$r_{g+}\ =\ r_{g-}\ =\ \frac{L_{c\ell\ min}}{\sqrt{2}} = L_{s\ max} = \frac{\ell_s}{2^{1/4}} = 0.841\ \ell_s \quad (72)$$

$(r_{g\ min\ bhAdS} = 2.304\ r_{g\ min\ bhdS})$

$$c\sqrt{\frac{|\Lambda|}{3}} = \frac{c}{l_s} 0.841 \qquad (D=4) \qquad (73)$$

KERR NEWMAN BH ($J \neq 0$, $Q \neq 0$)

$$T_{sem}(J,Q) = \frac{\hbar c}{2\pi k_B} \frac{1}{l_{c\ell}(J,Q)} \tag{74}$$

$$l_{c\ell}(J,Q) = \frac{2\ l_{c\ell}}{\delta}\left(1\ +\ \delta\ -\ \frac{\nu^2}{2}\right) \tag{75}$$

$$\nu \equiv \frac{l_Q}{l_{c\ell}} \quad , \quad \mu \equiv \frac{l_J}{l_{c\ell}} \quad , \quad \delta \equiv \sqrt{1\ -\ \nu^2\ -\ \mu^2} \tag{76}$$

$$l_{c\ell} = \frac{GM}{c^2}, \quad l_J = \frac{J}{Mc}, \quad l_Q = \frac{\sqrt{G}Q}{c^2} \tag{77}$$

$$l_{c\ell}^2 \geq l_Q^2 + l_J^2 \tag{78}$$

$(r_{\pm} = l_{c\ell} \pm (l_{c\ell}^2 - l_Q^2 - l_J^2)^{1/2})$

$$\nu^2\ +\ \mu^2 \leq 1 \tag{79}$$

$$\mu^2 \leq 1 \quad , \quad \nu^2 \leq 1 \tag{80}$$

$(\nu^2\ +\ \mu^2 = 1;\ T_{sem}(J,Q)_{extremal} = 0)$

$$T_{sem}(J,Q) \leq T_{sem}(J = 0 = Q) \tag{81}$$

$(l_{c\ell}(J,Q) \geq 4\ l_{c\ell} \equiv 2r_g)$

$$ln Z, \sigma_{string} \sim \frac{1}{t_s - T_{sem}} \qquad (T_{sem} \to t_s) \tag{82}$$

$$t_s = \frac{\hbar c}{2\pi k_B} \frac{1}{\hat{l_s}} \tag{83}$$

$$\hat{l_s} = \frac{b}{2\pi} \sqrt{\frac{\hbar \alpha'}{c}} \equiv \frac{b}{2\pi} l_s \tag{84}$$

$$T_{sem}(J, Q) < t_s \tag{85}$$

$$l_{c\ell} \geq \hat{l_s} \tag{86}$$

$T_{sem} \to t_s$ (string limit)

(i) $l_{c\ell} > \hat{l_s}/4$: $J \neq 0, Q \neq 0 \to J = 0, Q = 0$

(ii)$l_{c\ell} = \hat{l_s}/4$: $J \neq 0, Q \neq 0 \to J = 0, Q = 0$; $J = 0, Q \neq 0 (J \neq 0, Q = 0) \to J = 0, Q = 0$

(iii)$l_{c\ell} < \hat{l_s}/4$):$J \neq 0, Q > Q_0 \to J = 0, Q = Q_0$; $(J > J_0)$

KERR BH ($J \neq 0$, $Q = 0$)

$$S_{sem}(M,J) = S^{(0)}_{sem}(M,J) - a\ k_B\ \ln\left(\frac{S^{(0)}_{sem}(J,M)}{k_B} \right) + k_B\ \ln\ F(S^{(0)}_{sem}, J) \quad (87)$$

$$S^{(0)}_{sem}(J) = \frac{1}{2}(1+\Delta) S^{(0)}_{sem}(J=0) \quad (88)$$

$$\Delta = \sqrt{1 - \left(\frac{l_J}{l_{c\ell}}\right)^2} \quad (89)$$

$$S^{(0)}_{sem}(J=0) = \frac{1}{2}\frac{Mc^2}{T_{sem}(J=0)} \quad (90)$$

$$T_{sem}(J=0) = \frac{\hbar c}{2\pi k_B}\frac{1}{4L_{cl}} \quad (91)$$

(extremal $KerrBH$: $J = M^2 G/c^2$)

$$S_{sem}(M,J)_{extremal} \simeq -k_B\ \ln\left(\sqrt{\frac{2}{M}}\sqrt{M - (\frac{J}{\hbar})^{1/2} m_{Pl}}\right) \quad (92)$$