Formation of large-scale structures in the Universe: non-linear regime

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Introduction

Introduction

The Universe is homogeneous and isotropic at large scales (CMB) but it displays intricate structures at small scales: galaxies, clusters, filaments, voids,..

In usual scenarios, these highly non-linear density fluctuations have formed through the amplification by gravitational instability of small primordial density fluctuations, generated for instance during an inflationary phase. Besides, in the simplest cases these initial fluctuations are Gaussian and their amplitude increases at smaller scales. Therefore, smaller scales turn non-linear first and small objects merge to build increasingly large objects (galaxies, clusters of galaxies,...), following a hierarchical scenario.

In addition, a large fraction of the matter content of the Universe is made of collisionless dark matter particles ($\Omega_{\rm dm}/\Omega_{\rm b} \sim 7$).

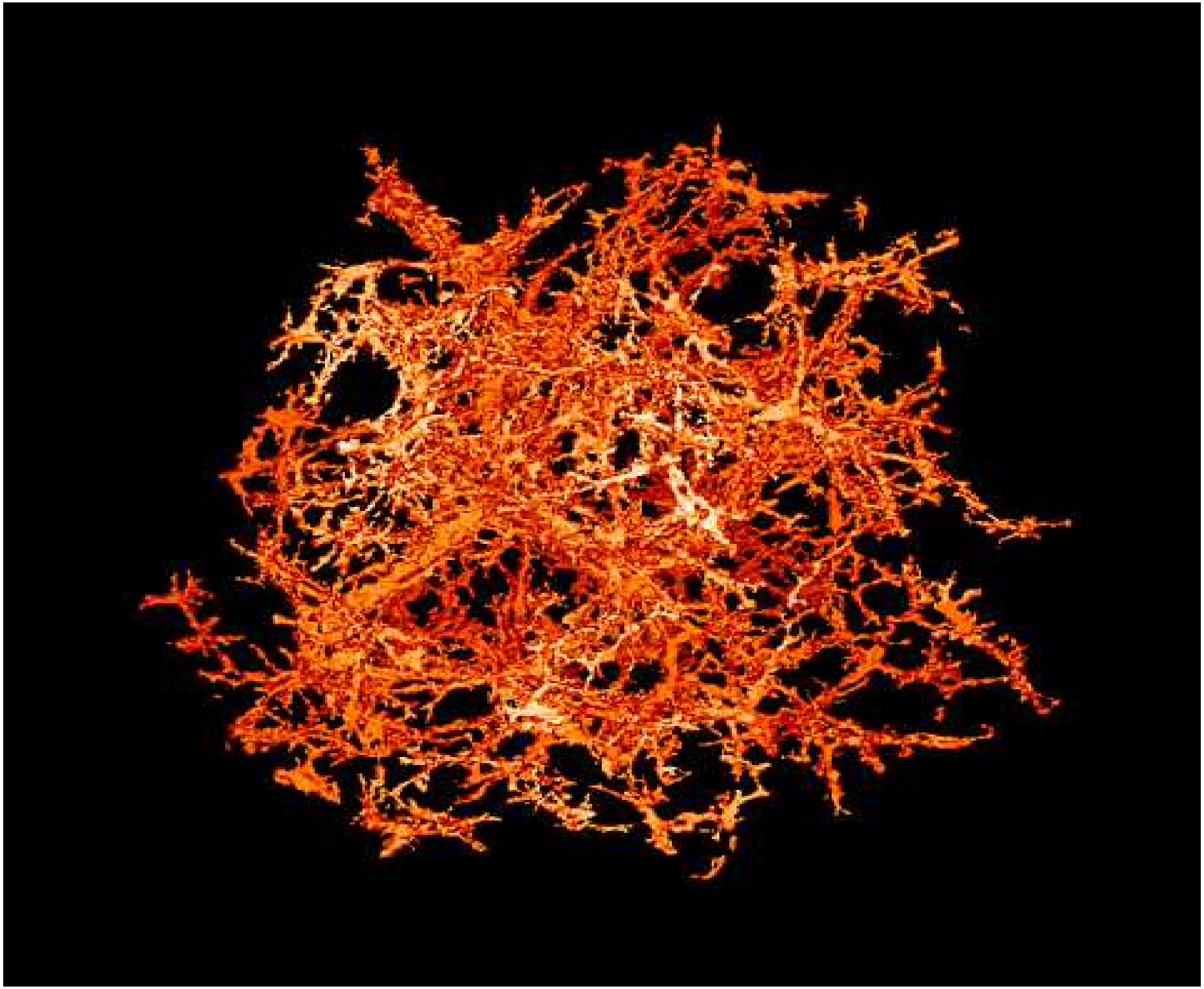
Introduction

On large scales collisional effects can be neglected and at scales much smaller than the horizon (and for small potentials) the Newtonian approximation is valid.

However, the Newtonian gravitational dynamics of collisionless particles in an expanding background is still a difficult problem: out-of-equilibrium dynamics.

- Linear regime: study of the linear growing (and decaying) modes.
- Quasi-linear regime: perturbative expansion over powers of the small initial fluctuations. Pb: the expansion is not well-behaved.
- N-body simulations. Pb: computational cost, physical insight.
- Phenomenological descriptions: Halo model, hierarchical models. Pb: not accurate enough for precision cosmology.

Introduction



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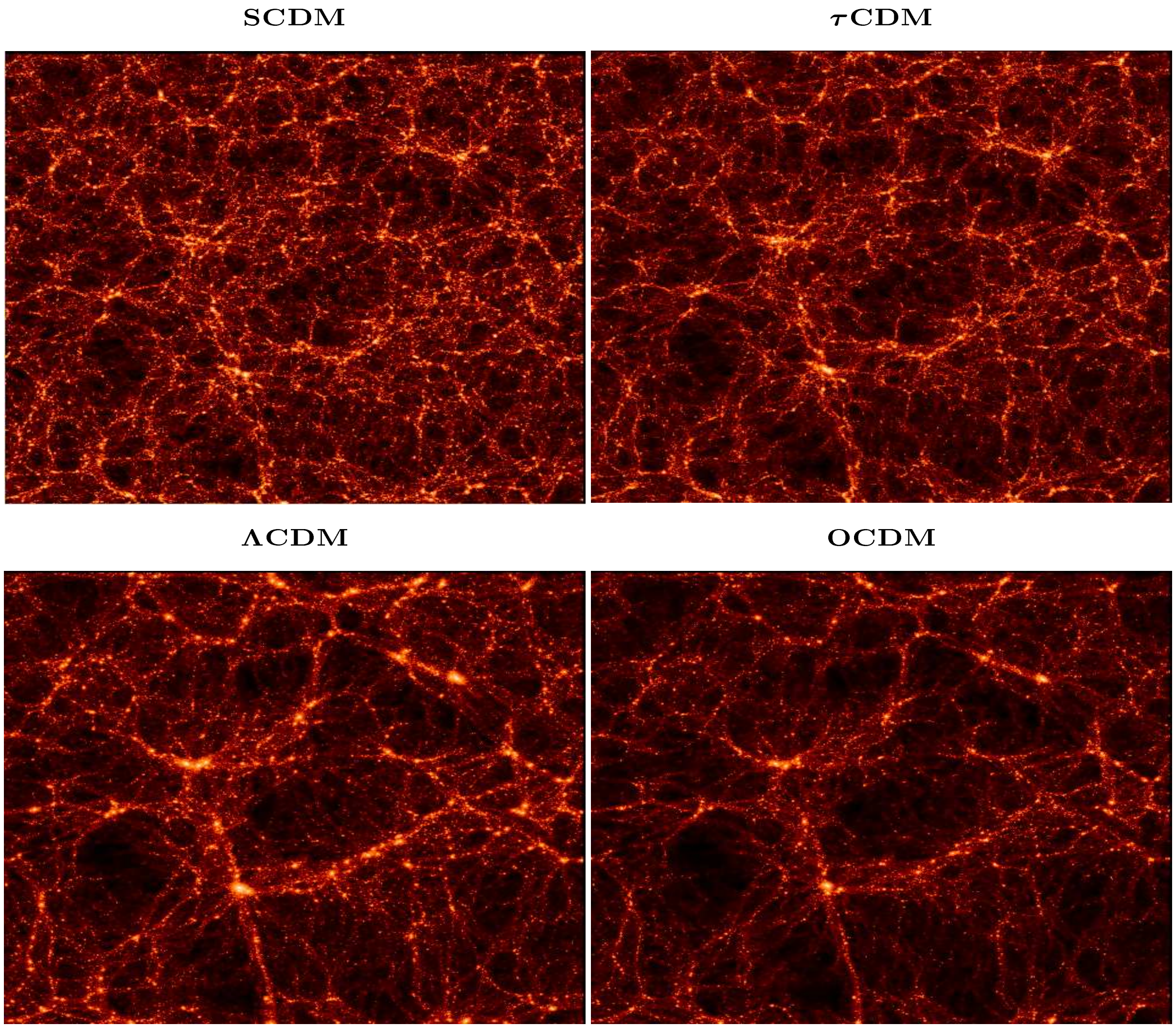


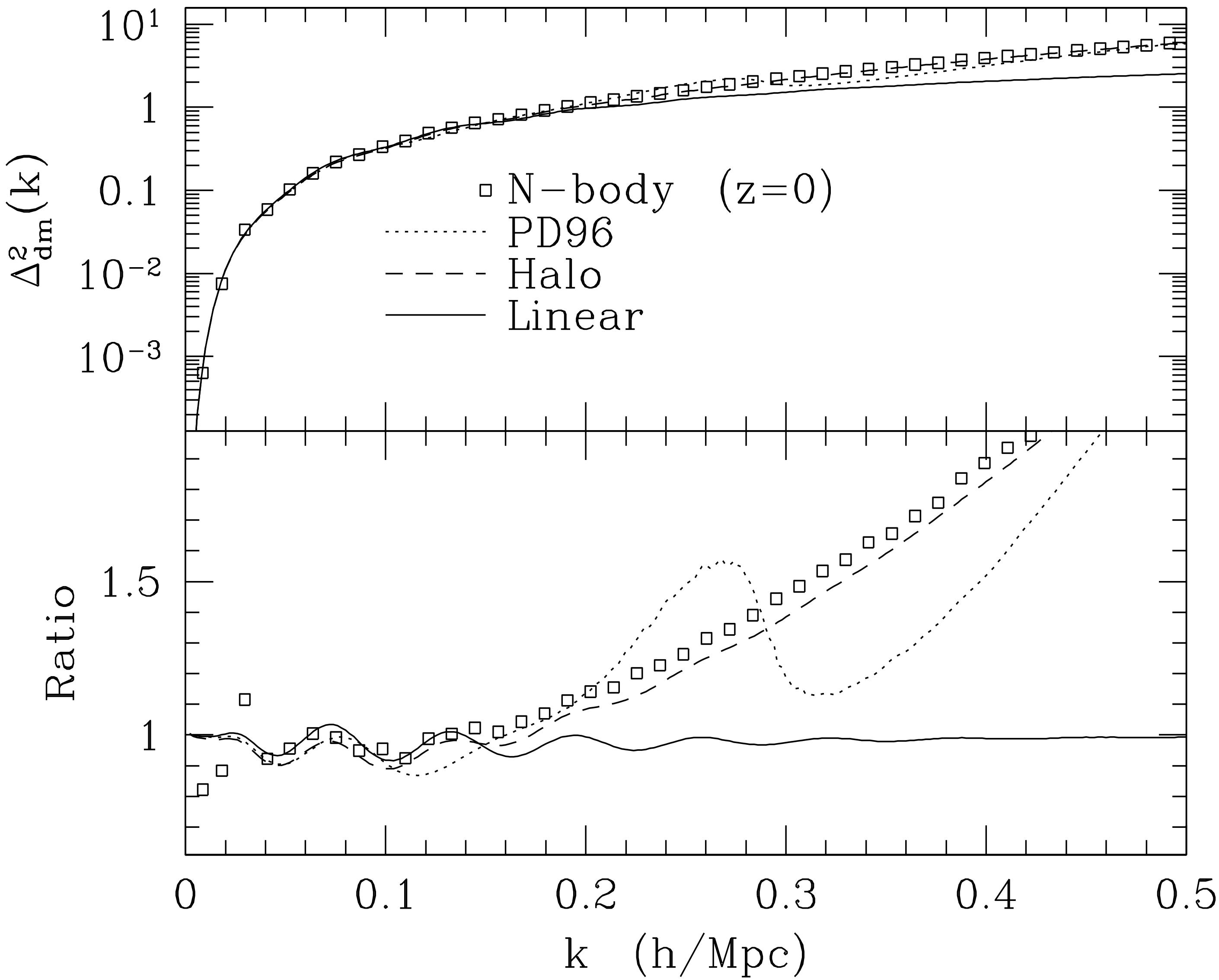
The VIRGO Collaboration 1996

Introduction

A good description of weakly non-linear scales is becoming of great practical interest as it can be a limiting factor for the accuracy of cosmological probes used to constrain the cosmological parameters. For instance, it is required to study:

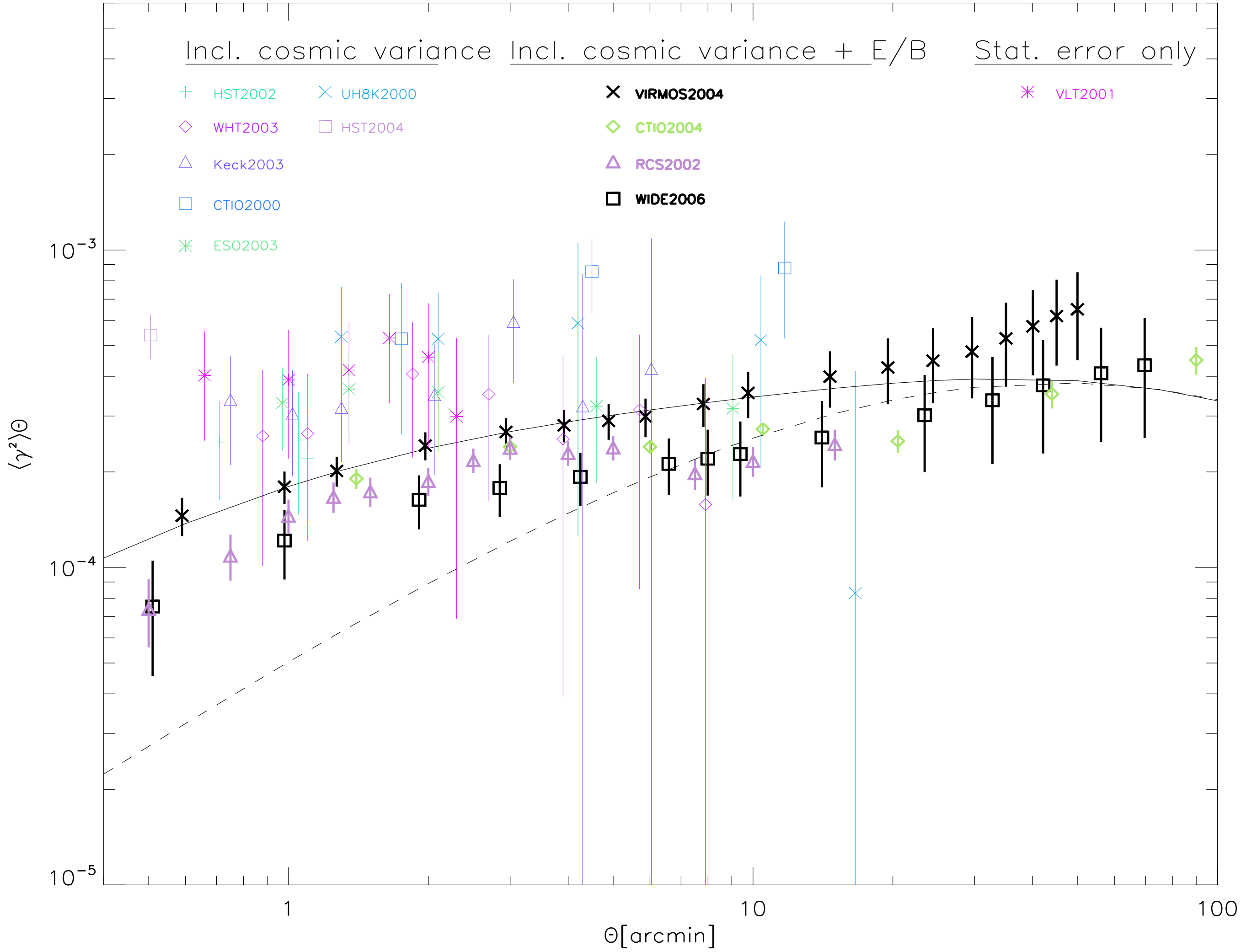
- Baryon acoustic oscillations
- Weak gravitational lensing distortions of distant galaxies.
- Transition linear/non-linear: can be used to constrain cosmological parameters through the dependence on the growth factor.

Introduction



The dark matter power spectrum from one simulation along with two common ansätze. [Huff et al. (2007)]

Introduction



The vertical axis is the shear top-hat variance multiplied by the angular scale in arcminutes. The horizontal axis is the radius of the smoothing window. [Munshi et al. (2006)]

Perturbative expansion

Linear regime

At large scales or at early times the fluctuations with respect to the homogeneous Hubble flow are small and we can linearize the equations of motion.

$$\begin{cases} \frac{\partial \delta}{\partial \tau} + \theta = 0 \\ \frac{\partial \theta}{\partial \tau} + \mathcal{H}\theta + \frac{3}{2}\Omega_{\rm m}\mathcal{H}^2\delta = 0 \\ \frac{\partial \mathbf{w}}{\partial \tau} + \mathcal{H}\mathbf{w} = 0 \end{cases} \quad \text{with} \quad \begin{cases} \theta = \nabla.\mathbf{v} \\ \mathbf{w} = \nabla \times \mathbf{v} \end{cases} \quad \tau = \int \frac{\mathrm{d}t}{a}.$$

This yields a growing and a decaying mode, with a potential velocity field:

$$\delta_L(\mathbf{x},\tau) = D_+(\tau)A(\mathbf{x}) + D_-(\tau)B(\mathbf{x}),$$

$$\frac{\mathrm{d}^2 D}{\mathrm{d}\tau^2} + \mathcal{H}\frac{\mathrm{d}D}{\mathrm{d}\tau} = \frac{3}{2}\Omega_{\rm m}\mathcal{H}^2 D.$$

For the Einstein-de Sitter Universe, $\Omega_{\rm m} = 1, \Omega_\Lambda = 0$, we have:

$$D_+ = a \propto t^{2/3}, \quad D_- = a^{-3/2} \propto t^{-1}.$$

Perturbative expansion

Of course, it is possible to go beyond the linear regime by performing a perturbative expansion. In this case, it is actually an expansion over powers of the linear growing mode. For the Einstein-de Sitter Universe each order can be factorized as:

$$\delta(\mathbf{x},\tau) = \sum_{n=1}^{\infty} a^n \delta_n(\mathbf{x})$$

$$\theta(\mathbf{x},\tau) = -\mathcal{H} \sum_{n=1}^{\infty} a^n \theta_n(\mathbf{x})$$

with:

$$\delta_n(\mathbf{x}) = \int \mathrm{d}\mathbf{x}_1..\mathrm{d}\mathbf{x}_n F_n(\mathbf{x};\mathbf{x}_1,..,\mathbf{x}_n)\delta_{L0}(\mathbf{x}_1)..\delta_{L0}(\mathbf{x}_n).$$

In the case of more general cosmologies, we can again obtain separable solutions with:

$$a \to D_+, \quad \text{if} \quad f = \frac{1}{\mathcal{H}D_+}\frac{\mathrm{d}D_+}{\mathrm{d}\tau} \simeq 1.$$

Perturbative expansion

In practice, it is convenient to work in Fourier space. Thus we define:

$$\delta(\mathbf{x}) = \int \mathrm{d}\mathbf{k}\; e^{i\mathbf{k}.\mathbf{x}} \delta(\mathbf{k}), \quad \langle \delta(\mathbf{k}_1)\delta(\mathbf{k}_2)\rangle = \delta_D(\mathbf{k}_1+\mathbf{k}_2) P(k_1)$$

where $P(k)$ is the matter power-spectrum. Next, taking the average over the Gaussian linear growing mode one obtains an expansion for the non-linear power-spectrum $P(k)$ over powers of the linear power-spectrum $P_{L0}(k)$.

However, this perturbative expansion is not very well behaved.

- Small-scale divergences.
- The higher-order terms are increasingly large and the series does not converge well:

$$P(k;a) = D_+^2 P_{L0}(k) + D_+^4 P^{(2)}(k) + D_+^6 P^{(3)}(k) + \ldots$$

Perturbative expansion

In the Zeldovich approximation one obtains:

$$P(k) = \int \frac{\mathrm{d}\mathbf{x}}{(2\pi)^3} e^{i\mathbf{k}.\mathbf{x}} \left[e^{-[k^2\sigma_v^2 - I(\mathbf{k},\mathbf{x})]} - 1 \right],$$

with $I(\mathbf{k},\mathbf{x}) = \int \mathrm{d}\mathbf{q} e^{i\mathbf{q}.\mathbf{x}} \frac{(\mathbf{k}.\mathbf{q})^2}{q^4} P_L(q), \quad \sigma_v^2 = \frac{I(k,0)}{k^2}.$

σ_v is the variance of the displacement field (and also the one-dimensional velocity dispersion in linear theory).

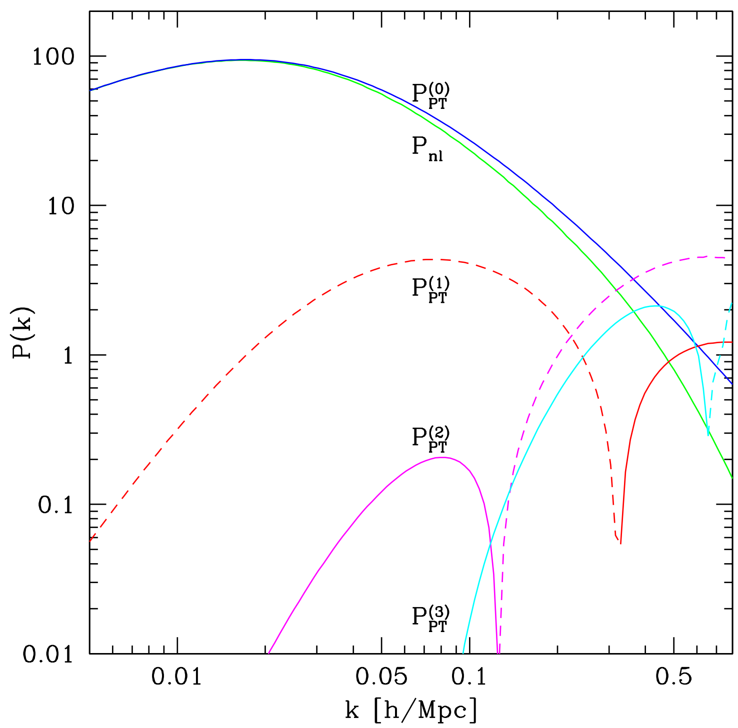
$$P(k) = \int \frac{\mathrm{d}\mathbf{x}}{(2\pi)^3} e^{i\mathbf{k}.\mathbf{x}} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} [k^2\sigma_v^2 - I(\mathbf{k},\mathbf{x})]^n \equiv \sum_{n=1}^{\infty} P_{\mathrm{PT}}^{(n)}(k)$$

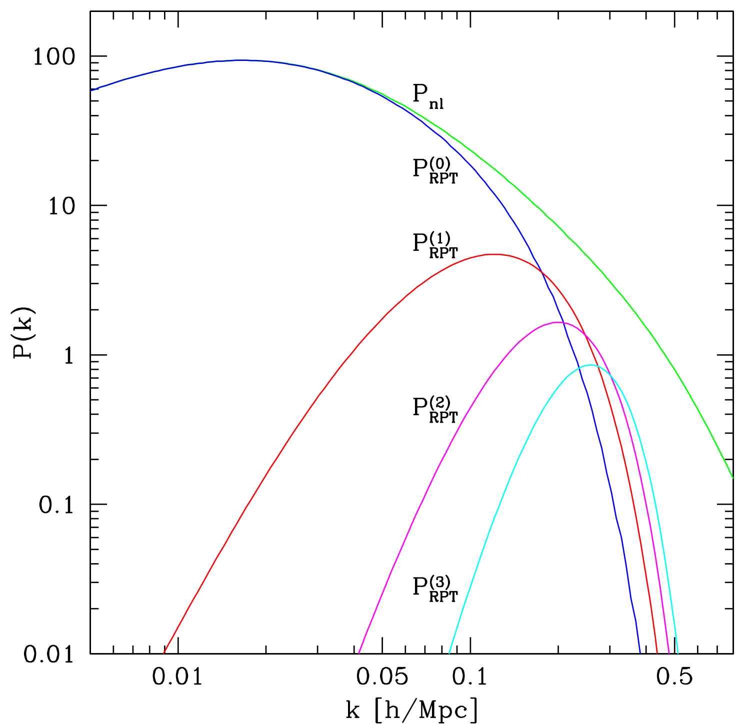
Crocce & Scoccimarro (2006) noticed that one can reorganize the perturbative expansion to obtain a well-behaved series:

$$P(k) = \int \frac{\mathrm{d}\mathbf{x}}{(2\pi)^3} e^{i\mathbf{k}.\mathbf{x}} e^{-k^2\sigma_v^2} \sum_{n=1}^{\infty} \frac{[I(\mathbf{k},\mathbf{x})]^n}{n!} \equiv \sum_{n=1}^{\infty} P_{\mathrm{RPT}}^{(n)}(k)$$

$$\langle \frac{\mathcal{D}\delta(\boldsymbol{k})}{\mathcal{D}\delta_L(\boldsymbol{k})} \rangle = D_+ \exp(-k^2\sigma_v^2/2)$$

Perturbative expansion





Comparison between PT (left) and RPT (right) expansions in the Zel'dovich approximation. Dashed lines denote negative values. [Crocce & Scoccimarro 2006]

A path-integral formalism

Equations of motion

We can first rewrite the equations of motion in a more concise form in Fourier space by using the Poisson eq. into the Euler eq.

$$\frac{\partial\delta(\mathbf{k},\tau)}{\partial\tau}+\theta(\mathbf{k},\tau) = -\int \mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\ \delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\alpha(\mathbf{k}_1,\mathbf{k}_2)\theta(\mathbf{k}_1,\tau)\delta(\mathbf{k}_2,\tau)$$

$$\frac{\partial\theta}{\partial\tau}+\mathcal{H}\theta+\frac{3}{2}\Omega_{\mathrm{m}}\mathcal{H}^2\delta = -\int \mathrm{d}\mathbf{k}_1\mathrm{d}\mathbf{k}_2\ \delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\beta(\mathbf{k}_1,\mathbf{k}_2)\theta(\mathbf{k}_1,\tau)\theta(\mathbf{k}_2,\tau)$$

with the mode-coupling vertices:

$$\alpha(\mathbf{k}_1,\mathbf{k}_2) = \frac{(\mathbf{k}_1+\mathbf{k}_2).\mathbf{k}_1}{k_1^2}, \quad \beta(\mathbf{k}_1,\mathbf{k}_2) = \frac{|\mathbf{k}_1+\mathbf{k}_2|^2(\mathbf{k}_1.\mathbf{k}_2)}{2k_1^2k_2^2}.$$

Next, it is convenient to define the two-component field ψ_i:

$$\psi(\mathbf{k},\eta) = \begin{pmatrix} \psi_1(\mathbf{k},\eta) \\ \psi_2(\mathbf{k},\eta) \end{pmatrix} = \begin{pmatrix} \delta(\mathbf{k},\eta) \\ -\theta(\mathbf{k},\eta)/\mathcal{H}f \end{pmatrix} \quad \text{with} \quad \eta = \ln\frac{D_+(\tau)}{D_{+0}}.$$

Equations of motion

Then, the equations of motion can be written as:

$$\mathcal{O}(x,x').\psi(x') = K_s(x;x_1,x_2).\psi(x_1)\psi(x_2) \quad \text{with} \quad x = (\mathbf{k},\eta,i).$$

The matrix $\mathcal{O}$ reads:

$$\mathcal{O}(x,x') = \begin{pmatrix} \frac{\partial}{\partial\eta} & -1 \\ -\frac{3\Omega_{\mathrm{m}}}{2f^2} & \frac{\partial}{\partial\eta} + \frac{3\Omega_{\mathrm{m}}}{2f^2} - 1 \end{pmatrix} \delta_D(\mathbf{k}-\mathbf{k}')\delta_D(\eta-\eta')$$

whereas the symmetric vertex $K_s(x;x_1,x_2) = K_s(x;x_2,x_1)$ writes:

$$K_s(x;x_1,x_2) = \delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\delta_D(\eta_1-\eta)\delta_D(\eta_2-\eta)\gamma^s_{i;i_1,i_2}(\mathbf{k}_1,\mathbf{k}_2)$$

with:

$$\gamma^s_{1;1,2}(\mathbf{k}_1,\mathbf{k}_2) = \frac{\alpha(\mathbf{k}_2,\mathbf{k}_1)}{2}, \quad \gamma^s_{1;2,1}(\mathbf{k}_1,\mathbf{k}_2) = \frac{\alpha(\mathbf{k}_1,\mathbf{k}_2)}{2},$$

$$\gamma^s_{2;2,2}(\mathbf{k}_1,\mathbf{k}_2) = \beta(\mathbf{k}_1,\mathbf{k}_2), \quad \text{and zero otherwise .}$$

Action $S[\psi, \lambda]$

Since we are only interested in the statistical properties of the system it is convenient to apply a path-integral formalism. To do so, we can write:

$$\mathcal{O}.\psi = K_s.\psi\psi + \mu_i \quad \text{with} \quad \mu_i(x) = \delta_D(\eta - \eta_i) e^{\eta_i} \delta_{L0}(\mathbf{k}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

and $\psi = 0$ for $\eta < \eta_i$. The source μ_i provides the initial conditions, eventually we let $\eta_i \to -\infty$.
Then, we define the generating functional $Z[j]$:

$$Z[j] = \langle e^{j.\psi} \rangle = \int [\mathrm{d}\mu_i] \, e^{j.\psi[\mu_i] - \frac{1}{2}\mu_i.\Delta_i^{-1}.\mu_i}$$

$$\text{with} \quad \langle \mu_i \rangle = 0, \quad \langle \mu_i(x_1)\mu_i(x_2) \rangle = \Delta_i(x_1, x_2).$$

This also reads:

$$Z[j] = \int [\mathrm{d}\mu_i][\mathrm{d}\psi] \, |\det M| \, \delta_D(\mu_i - \mathcal{O}.\psi + K_s.\psi\psi) \, e^{j.\psi - \frac{1}{2}\mu_i.\Delta_i^{-1}.\mu_i}$$

Action $S[\psi, \lambda]$

Writing the Dirac functional as an integral we obtain:

$$Z[j] = \int [\mathrm{d}\psi][\mathrm{d}\lambda]\; e^{j.\psi + \lambda.(-\mathcal{O}.\psi + K_s.\psi\psi) + \frac{1}{2}\lambda.\Delta_i.\lambda}$$

Therefore, the system is described by the action $S[\psi, \lambda]$:

$$S[\psi, \lambda] = \lambda.(\mathcal{O}.\psi - K_s.\psi\psi) - \frac{1}{2}\lambda.\Delta_i.\lambda.$$

We are interested in the two-point correlation $G(x_1, x_2)$ and response function $R(x_1, x_2)$ defined as:

$$G(x_1, x_2) = \langle \psi(x_1)\psi(x_2) \rangle, \quad R(x_1, x_2) = \langle \frac{\delta\psi(x_1)}{\delta\zeta(x_2)} \rangle_{\zeta=0}$$

and $\; R(x_1, x_2) \propto \theta(\eta_1 - \eta_2), \quad \eta_1 \to \eta_2 : \quad R(x_1, x_2) \to \delta_D(\mathbf{k}_1 - \mathbf{k}_2)\delta_{i_1, i_2}.$

Moreover, the auxiliary field λ allows us to obtain R through:

$$R(x_1, x_2) = \langle \psi(x_1)\lambda(x_2) \rangle, \quad \langle \lambda \rangle = 0, \quad \langle \lambda\lambda \rangle = 0.$$

Large-N-expansions

In order to evaluate this path-integral it is possible to investigate large-N expansions:

$$Z_N[j,h] = \int [\mathrm{d}\psi][\mathrm{d}\lambda]\; e^{N[j.\psi + h.\lambda - S[\psi,\lambda]]}$$

I. Direct steepest-descent method: This yields for auxiliary correlation and response functions G_0 and R_0:

$$\mathcal{O}(x,z).G_0(z,y) = 0, \quad \mathcal{O}(x,z).R_0(z,y) = \delta_D(x-y),$$

whereas the actual correlation and response functions obey:

$$\mathcal{O}(x,z).G(z,y) = \Sigma(x,z).G(z,y) + \Pi(x,z).R^T(z,y)$$

$$\mathcal{O}(x,z).R(z,y) = \delta_D(x-y) + \Sigma(x,z).R(z,y)$$

We took the limit $\eta_i \to -\infty$ so that terms involving Δ_i vanish. We can see that the auxiliary matrices G_0 and R_0 are actually equal to their linear counterparts: $G_0 = G_L, R_0 = R_L$.

Large-N-expansions

The self-energy terms Σ and Π are given at one-loop order by:

$$\Sigma(x,y) = 4K_s(x;x_1,x_2)K_s(z;y,z_2)R_0(x_1,z)G_0(x_2,z_2)$$

$$\Pi(x,y) = 2K_s(x;x_1,x_2)K_s(y;y_1,y_2)G_0(x_1,y_1)G_0(x_2,y_2)$$

The expansion over powers of $1/N$ only enters the expression of the self-energy.

II. 2PI effective action method: This yields the same equations where G_0 and R_0 are replaced by the non-linear two-point functions G and R in the self-energy.

Thus, the the direct steepest-descent method yields a series of linear equations which can be solved directly whereas the 2PI effective action method gives a system of non-linear equations which must usually be solved numerically by an iterative scheme. However, thanks to the Heaviside factors appearing in the response R and the self-energy Σ these equations can be solved directly by integrating forward over time η_1.

General properties

- Agreement with the standard perturbative analysis over powers of $P_{L0}(k)$ up to the order used for the self-energy. As compared with the standard perturbative approach, the two schemes described above also include two different infinite partial resummations.
- The equations obtained for the hydrodynamical system are simpler than for the collisionless system described by the Vlasov-Poisson system.
- The correlation G can also be written as:

$$G(x_1, x_2) = R \times G_0(\eta_i) \times R^T + R.\Pi.R^T$$

- Using a diagrammatic technique Crocce & Scoccimarro (2006) derived these Schwinger-Dyson equations in an integral form.

Direct steepest-descent method

Response function

The evolution equations for the response function R read:

$$\frac{\partial R_1}{\partial \eta_1} - R_2 = \Sigma_{0;11}.R_1 + \Sigma_{0;12}.R_2$$

$$\frac{\partial R_2}{\partial \eta_1} - \frac{3}{2}R_1 + \frac{1}{2}R_2 = \Sigma_{0;21}.R_1 + \Sigma_{0;22}.R_2$$

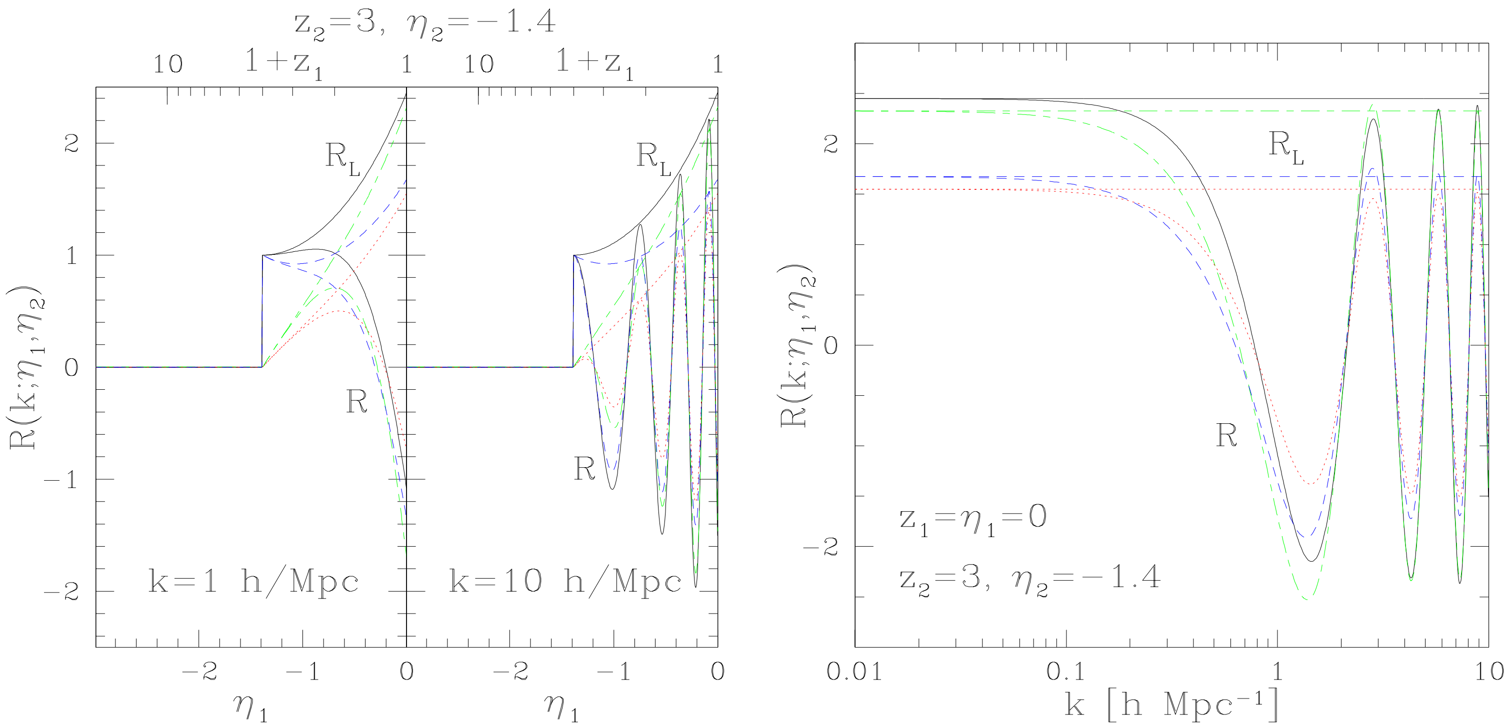
where $(R_1, R_2) = (R_{11}, R_{21})$ or (R_{12}, R_{22}). Taking advantage of the simple dependence on time of the r.h.s. it is possible to eliminate the time-integrals to obtain two coupled differential equations for R_1 and R_2. In the small-scale limit $|\Sigma_0^+| \sim k^2 \rightarrow \infty$ we obtain the asymptotic solution:

$$R(x_1, x_2) = R_L(x_1, x_2) \cos[\omega(a_1 - a_2)] + \mathcal{O}(1/\omega) \quad \text{with} \quad \omega = k\sigma_v .$$

Thus the direct steepest-descent method has given rise to a "UV cutoff" in the form of fast oscillations. The usual 1-loop result would be in the same small-scale limit:

$$R_{1-\mathrm{loop}}(x_1, x_2) = R_L(x_1, x_2) \left[1 - \frac{1}{2}\omega^2 (a_1 - a_2)^2 \right] .$$

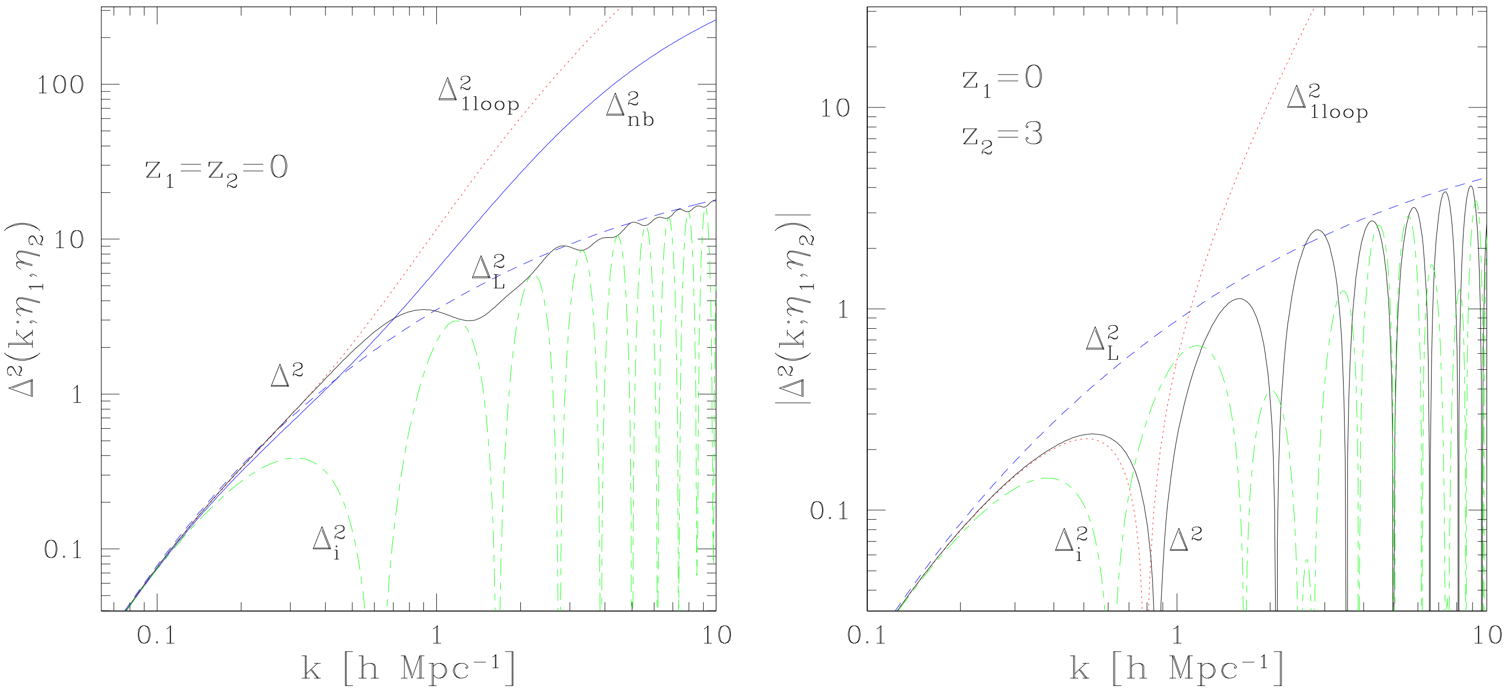
Response function



The response function $R_{i_1 i_2}(k;\eta_1,\eta_2)$

The non-linear response exhibits oscillations $\sim \cos(\omega a_1)$ with a frequency $\omega \sim k$ and an amplitude which follows the linear response R_L. Thus there is no true damping at this order, except after integration over time. Nevertheless, this is already an improvement over the standard perturbative expansion.

Correlation function



The power $\Delta^2(k; z_1, z_2)$.

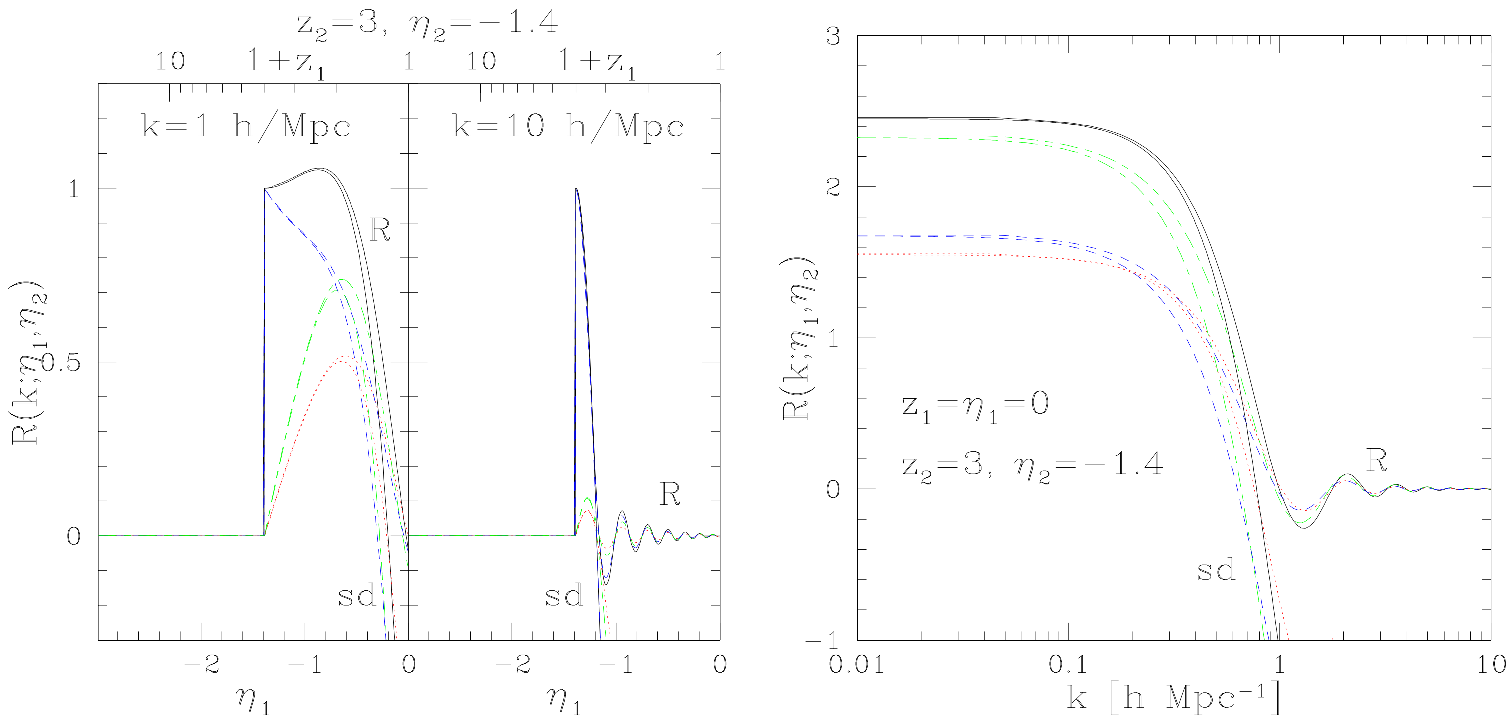
$$\Delta^2(k;\eta_1,\eta_2) = 4\pi k^3 G_{11}(k;\eta_1,\eta_2)$$

$$G(x_1,x_2) = R \times G_0(\eta_i) \times R^T + R.\Pi.R^T$$

The correlation G is better behaved in the highly non-linear regime than for the standard 1-loop result.

2PI effective action approach

Response function



The response function $R_{i_1 i_2}(k;\eta_1,\eta_2)$.

The non-linear response exhibits damped oscillations in the non-linear regime.

Response function

The behavior of the response function can be understood from the following simple model:

$$\frac{\partial R}{\partial \eta_1} = \sigma \int_{\eta_2}^{\eta_1} \mathrm{d}\eta \, e^{\eta_1+\eta} R(\eta_1,\eta) R(\eta,\eta_2).$$

The parameter $\sigma \leq 0$ represents the amplitude of the self-energy Σ at the wavenumber of interest.

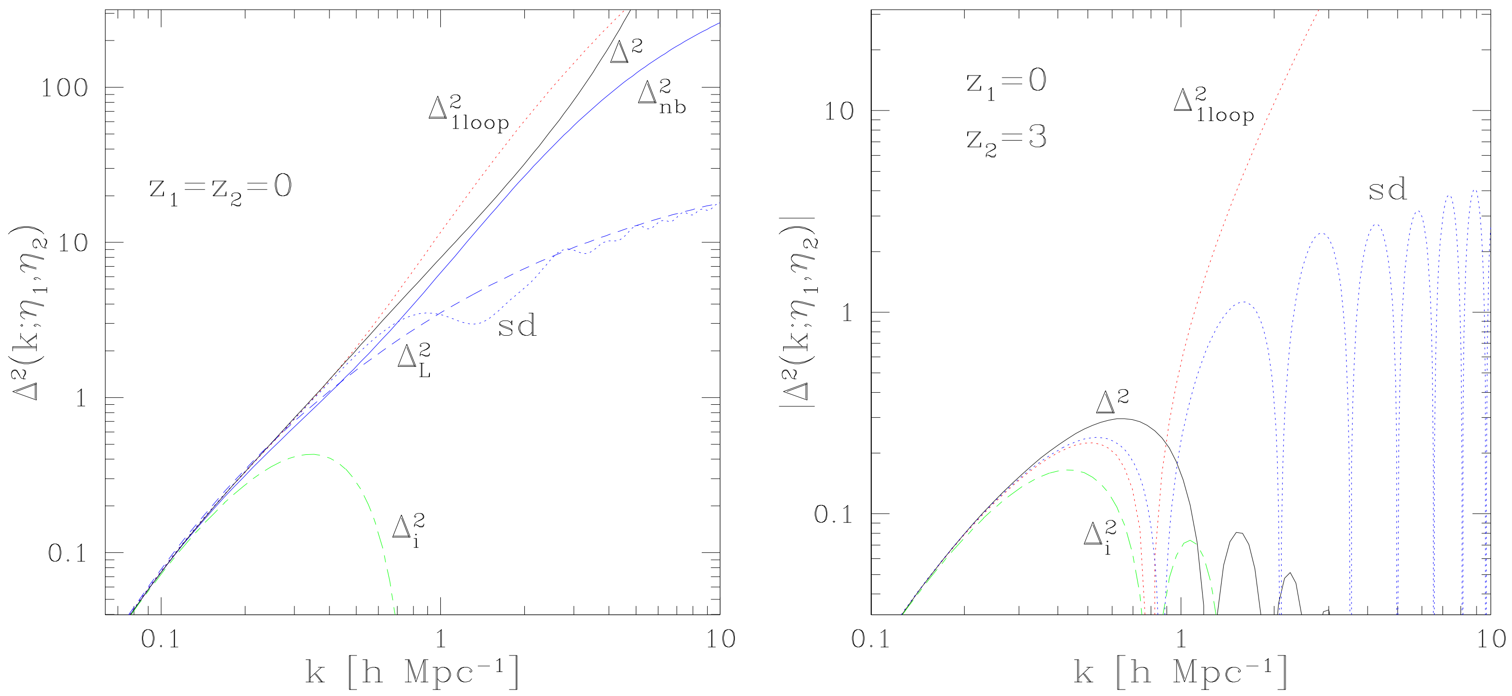
- Linear regime: $\sigma = 0, \quad R_L(\eta_1,\eta_2) = 1$.
- Steepest-descent method: $R(\eta_1,\eta) \rightarrow R_L(\eta_1,\eta)$. This linear equation yields:

$$\frac{\partial^2 R}{\partial a_1^2} = \sigma R, \quad R(\eta_1,\eta_2) = \cos[\omega(a_1-a_2)] \quad \text{with} \quad \omega = \sqrt{-\sigma}.$$

- 2PI effective action: The non-linear equation reads:

$$\frac{\partial R}{\partial a_1} = \sigma \int_{a_2}^{a_1} \mathrm{d}a \, R(a_1,a) R(a,a_2), \quad R(\eta_1,\eta_2) = \frac{J_1[2\omega(a_1-a_2)]}{\omega(a_1-a_2)}.$$

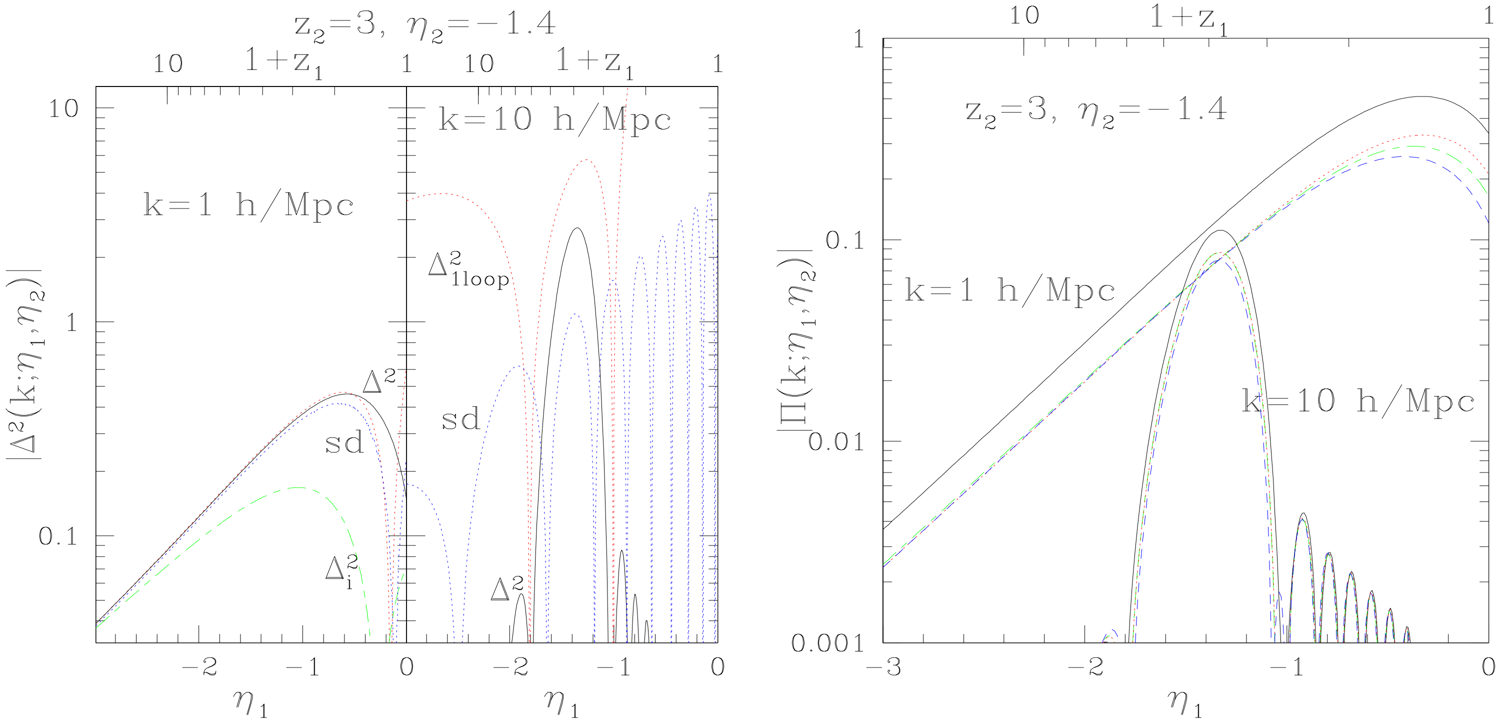
Correlation function



The power $\Delta^2(k; z_1, z_2)$.

The power Δ^2 keep growing at small scales for identical times but it is damped for unequal times.

Correlation function



The power $\Delta^2(k;\eta_1,\eta_2)$ and the self-energy Π as a function of time η_1.

The power peaks at equal times $\eta_1 = \eta_2$. There is a qualitative improvement over the standard perturbative expansion.

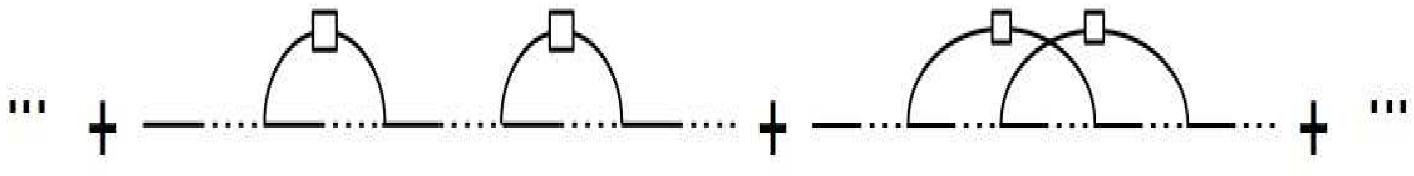
Using the high-k behavior of the response function R

Using the high-k Gaussian decay

In the large-k limit Crocce & Scoccimarro (2006) managed to resum a subset of diagrams using the property:

$$\gamma^s_{i;i_1,i_2}(\mathbf{q},\mathbf{k})\psi_{L,i_1} \simeq \frac{k}{2q}\cos(\mathbf{k}.\mathbf{q})\delta_{i,i_2}$$

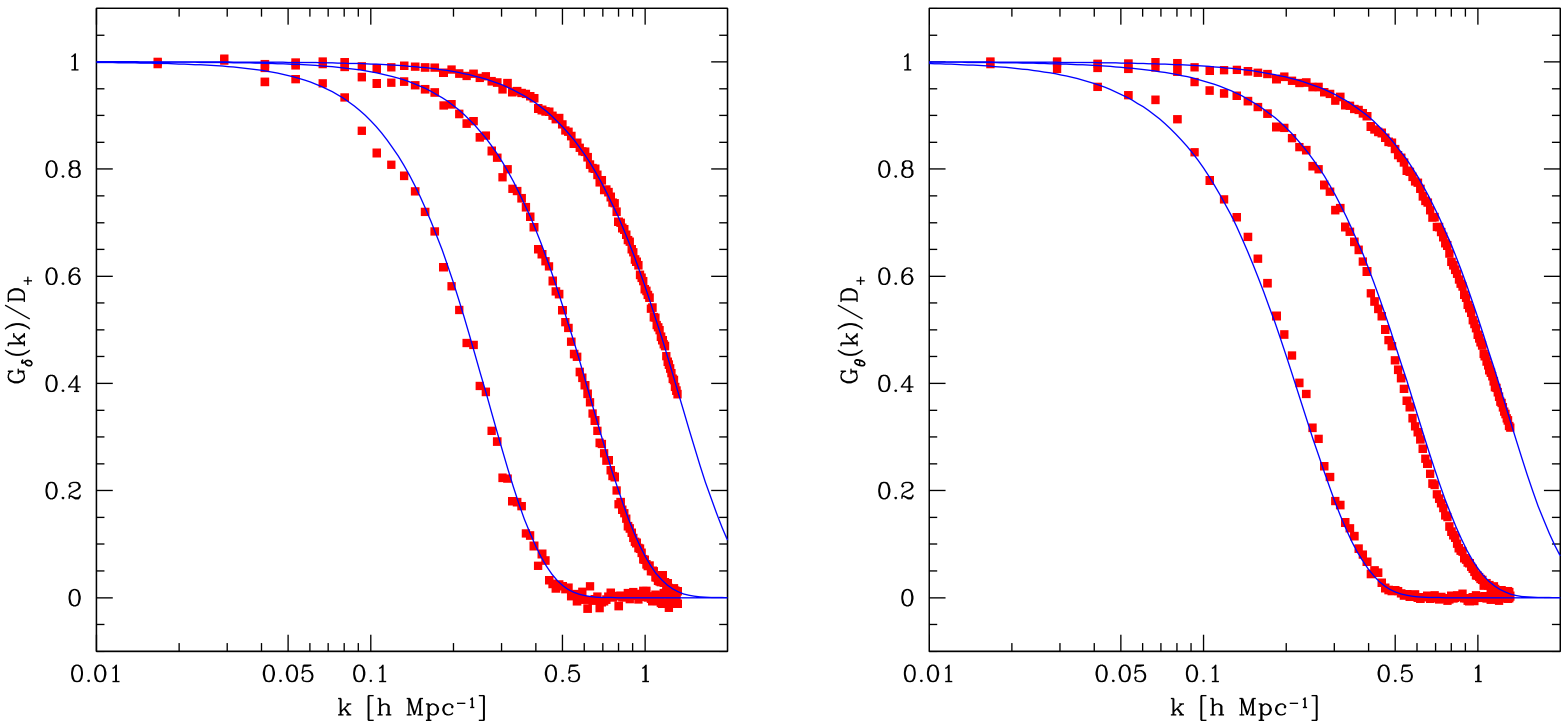
Then, resumming all diagrams of the form:



they obtained:

$$R(x_1,x_2) = R_L \; e^{-k^2\sigma_v^2(a_1-a_2)^2/2} .$$

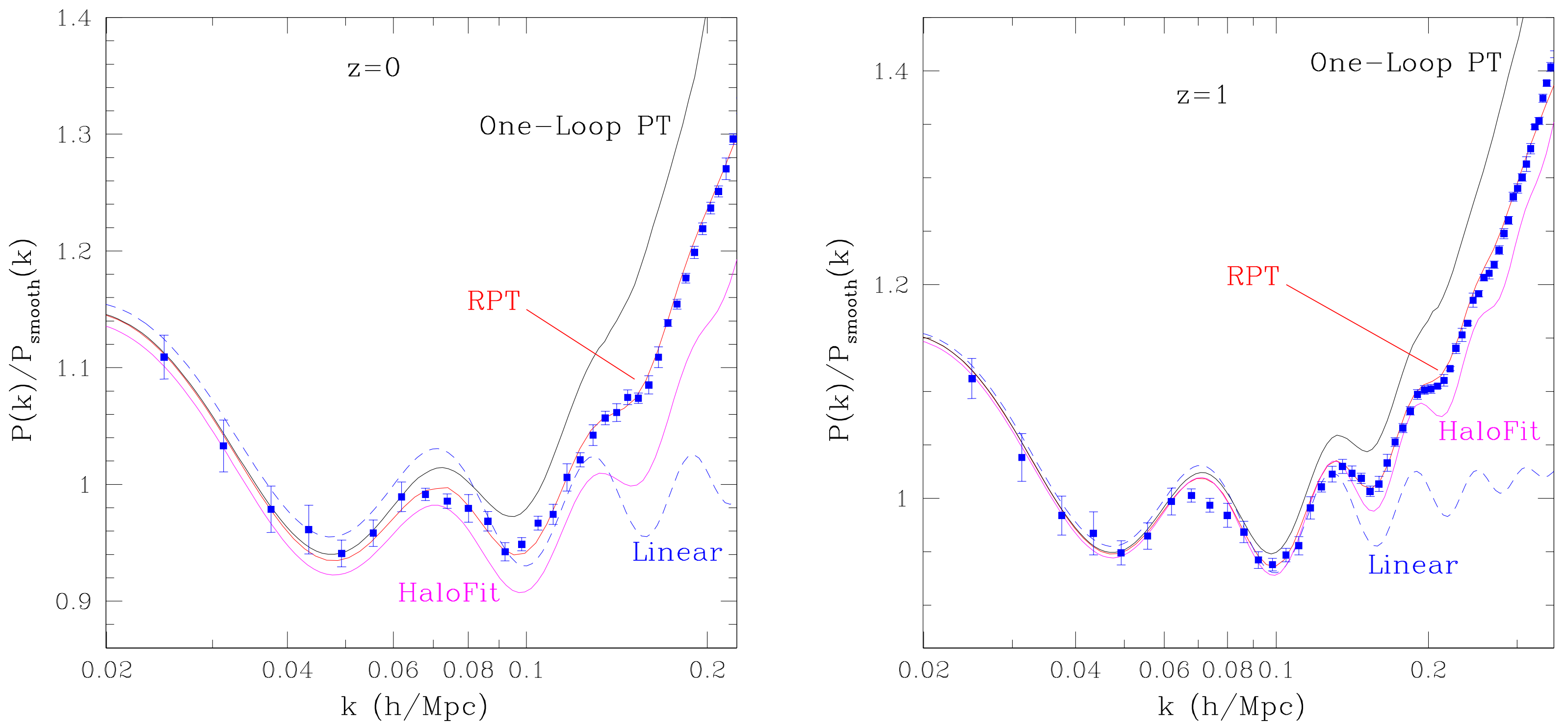
Using the high-k Gaussian decay



The response functions for the density and velocity divergence. The three cases correspond (from left to right in each panel) to $z = 0, 2, 5$. [Crocce & Scoccimarro (2006)].

Matching the 1-loop results obtained from standard perturbation theory to the high-k asymptotics provides a good fit to results from numerical simulations.

Using the high-k Gaussian decay



Nonlinear evolution of the acoustic oscillations in the dark matter power spectrum.. [Crocce & Scoccimarro (2007)].

One also obtains a good agreement with results from numerical simulations for the dark matter power-spectrum.

Conclusion

Conclusion

- Thus we have seen that the standard perturbative analysis of gravitational clustering in the expanding Universe can be improved by reorganizing the perturbative expansion. This can be performed in various ways:
    - Using a path-integral method and expansion schemes such as large-N expansions .
    - Resumming subsets of diagrams by looking at the high-k limit and to use the best model for the non-linear response function.
    - Looking at the evolution of the system as we take into account smaller scales.
- The result is better behaved than the standard perturbative expansion and the partial resummations give rise to a damping factor in the highly non-linear regime (instead of increasingly large powers).

Conclusion

- As checked by a comparison with numerical simulations, this provides better predictions over weakly non-linear scales. In particular, this is more reliable than phenomenological models such as the halo model. Note that these procedure apply to any cosmological models (including dynamical dark energy models).
  This can be used for baryon acoustic oscillations, weak-lensing studies..
- Some work still needs to be done:
  - Using more systematic procedures in some steps.
  - Going to higher orders than 1-loop diagrams.
  - Higher-order correlation functions such as the 3-point correlation (bispectrum).
  - Extension to the highly non-linear regime beyond the single-stream approximation.