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Dark Energy Is the Cosmological Quantum Vacuum Energy of Light Particles—The Axion and the Lightest Neutrino

Héctor J. de Vega [1,†] and Norma G. Sanchez [2,*,‡]

[1] CNRS LPTHE, Sorbonne Université, Université Pierre et Marie Curie UPMC, 75005 Paris, France
[2] CNRS PSL, Observatoire de Paris, Sorbonne Université and the Chalonge-de Vega International School Center, 75012 Paris, France
* Correspondence: norma.sanchez@obspm.fr
† passed away https://chalonge-devega.fr/HdeV.html.
‡ https://chalonge-devega.fr/sanchez/.

Abstract: We uncover the general mechanism and the nature of today's dark energy (DE). This is only based on well-known quantum physics and cosmology. We show that the observed DE today originates from the cosmological quantum vacuum of light particles, which provides a continuous energy distribution able to reproduce the data. Bosons give positive contributions to the DE, while fermions yield negative contributions. As usual in field theory, ultraviolet divergences are subtracted from the physical quantities. The subtractions respect the symmetries of the theory, and we normalize the physical quantities to be zero for the Minkowski vacuum. The resulting finite contributions to the energy density and the pressure from the quantum vacuum grow as $\log a(t)$, where $a(t)$ is the scale factor, while the particle contributions dilute as $1/a^3(t)$, as it must be for massive particles. We find the explicit dark energy equation of state of today to be $P = w(z)\ \mathcal{H}$: it turns to be slightly $w(z) < -1$ with $w(z)$ asymptotically reaching the value -1 from below. A scalar particle can produce the observed dark energy through its quantum cosmological vacuum provided that (i) its mass is of the order of 10^{-3} eV = 1 meV, (ii) it is very weakly coupled, and (iii) it is stable on the time scale of the age of the universe. The axion vacuum thus appears as a natural candidate. The neutrino vacuum (especially the lightest mass eigenstate) can give negative contributions to the dark energy. We find that $w(z = 0)$ is slightly below -1 by an amount ranging from (-1.5×10^{-3}) to (-8×10^{-3}) and we predict the axion mass to be in the range between 4 and 5 meV. We find that the universe will expand in the future faster than the de Sitter universe as an exponential in the square of the cosmic time. Dark energy today arises from the quantum vacuum of light particles in FRW cosmological space-time in an analogous way to the Casimir vacuum effect of quantum fields in Minkowski space-time with non-trivial boundary conditions.

Keywords: dark energy; cosmological quantum vacuum; axions; light meV neutrinos

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1. Introduction and Results

Since the discovery of dark energy in the present universe [1–4], intense observational activity has improved our knowledge about it [5–15], and more activity is expected to provide new data and understanding, e.g., [16,17]. Many different approaches and models have been proposed to explain dark energy [18–34]. For reviews on and approaches to dark energy, see, for example, refs. [18–34].

As is, by now, well known, let us mention that there exist current discordances between different cosmological probes, mainly the discrepancy in the value of the Hubble constant H_0: 5.0σ between early universe indirect H_0 determinations and late universe direct measurements of H_0. Regarding other stresses and anomalies of lower statistical significance, which are interesting in their own but are not the subject of this paper, see, for example, ref. [35] and references therein. As is well known too, there also exist theoretical discordances, such as the fine adjustment of the cosmological constant Λ, seen, for example, in refs. [22,34] and references therein. Clarification to this problem has been provided recently [36,37]: The huge difference between the observed value of Λ today and the particle physics-evaluated value Λ_Q is correct and must be physically like that because the two values correspond to the same physical magnitude but to two different vacuum states and cosmic eras—the observed Λ value today corresponds to the classical/semiclassical, large and dilute (mostly empty) universe of today, consistent with the very low observed Λ value (10^{-122} in Planck units), while the computed value Λ_Q (10^{+122} in Planck units) corresponds to the small, highly dense and energetic quantum gravity universe in its far (trans-Planckian) past, and this is consistent with its extremely high, trans-Planckian value. The two values are classical-quantum duals of each other in the sense of the classical-quantum (wave-particle) duality including gravity and independently agree with a path integral gravity derivation [36–38].

In this paper, we study the cosmological Quantum Field Theory (QFT) vacuum as dark energy within a fundamental analytic framework with explicit and analytic results, e.g., the derivation of the dark energy equation of state and the future evolution of the universe. Moreover, from these results, we also extract the implications and determination of the particles contributing to dark energy and compute their masses.

We show that the dark energy present today in the universe originates from the cosmological quantum vacuum of light particles in the meV mass scale. This is a vacuum effect that unavoidably appears when quantum fields evolve in a cosmological space-time. That is, dark energy today is generated by a mechanism based on well-known quantum physics and cosmology. Bosons yield positive contributions to the dark energy, while fermions give negative contributions.

We find that the scale of the contributions to the dark energy is of the order of

$$\frac{M^4}{2\,(4\pi)^2}\;\log z_{\,\mathrm{dec}}, \tag{1}$$

where M is the particle mass and z_{dec} is the redshift when it is decoupled from the early universe plasma.

Generally speaking, the energy of a quantum field is the sum of the vacuum contribution plus particle contributions. It is known that the vacuum energy of a quantum field dissipates into particles when the field evolves coupled to other fields or to itself [39–43]. Dissipation into fermions is reduced by Pauli blocking [41,42]. Electrons, protons and photons are coupled to photons and, therefore, their vacuum energy dissipates through photon production well before recombination, that is, when the temperature of the universe is 1 MeV or more. Unstable particles cannot produce long-lasting vacuum effects. Only a very weakly coupled stable particle can produce a vacuum energy contribution lasting for times of the order of the age of the universe, that is, a vacuum energy contribution measurable today.

Since dark energy is known to be positive, bosons must dominate the cosmological vacuum energy. The scale of the boson mass must be in the meV range because the observed dark energy density has the value [15,44–46]

$$\rho_\Lambda = \Omega_\Lambda \rho_c = (2.39\ \mathrm{meV})^4, \quad 1\ \mathrm{meV} = 10^{-3}\mathrm{eV}. \tag{2}$$

Spontaneous symmetry breaking of continuous symmetries is a natural way to produce massless scalars (Goldstone bosons) in particle physics. Furthermore, a slight violation of the corresponding symmetry can give a small mass to such a scalar particle. Axions, majorons and familons have been proposed on these grounds [47–56].

In addition, the lightest neutrino can give a negative contribution to dark energy.

Neutrinos are, by now, very well-motivated particles from the point of view of particle physics, cosmology and astrophysics, e.g., [57–59]. For Majorana-type neutrinos, neutrinos and antineutrinos coincide, while, for Dirac neutrinos, neutrinos and antineutrinos are distinct. It is not yet clear whether neutrinos are of Majorana or Dirac type, and, in this paper, we discuss the implications for dark energy of both of them. Interestingly enough, light meV neutrinos and the meV axion do appear here as a consequence of our results for the dark energy computed from first principles. For constraints on other types of neutrinos and other relativistic species or "dark radiation", see, for example, [60,61] and references therein.

Neutrinos in the universe are known to be free for temperatures $T \lesssim 1$ Mev, which correspond to redshifts $z \lesssim 6 \times 10^9$ [57–59]. That is, we can describe their evolution as free fermions in the cosmological FRW universe.

Axions with masses $M \sim 1$ meV are free for temperatures $T \lesssim 10^6$ GeV, which correspond to redshifts $z \lesssim 10^{19}$ [62–66]. They can be considered as free scalars in the cosmological FRW universe. Both the axion and neutrino decoupling happen during the radiation-dominated era. Before decoupling, the non-negligeable interaction of the corresponding particles made dissipation important, therefore the vacuum energy can only become significant after decoupling. Therefore, we can restrict ourselves to study free quantum field evolution in the cosmological space-time after decoupling.

- We investigate the evolution of scalars and fermions as an initial value problem (Cauchy problem) for the corresponding quantum fields on a cosmological space-time.
- We find that the initial temperature has a negligible effect on the vacuum energy for late times.
- Both axions and neutrinos can lead to vacuum effects lasting cosmological time scales. Any of the two heavier neutrino mass eigenstates, ν_2 and ν_3, would produce a large negative dark energy in the $(50 \text{ meV})^4$ range. Hence:
  (i) either the heavier neutrinos, ν_2 and ν_3, annihilate with their respective anti-neutrinos in a time scale of the age of the universe, or
  (ii) a stable scalar particle with mass in the $\gtrsim 50$ meV range must be present in order to reproduce the observed value of the dark energy Equation (2).

  However, we find in this paper that possibility (ii) is inconsistent with the observed dark energy equation of state.

An effective four-fermions interaction with strength characterized by M'^{-2}, where M' is a mass scale, can make the heavier neutrinos unstable. The mass scale M' should be $M' \lesssim 1$ MeV or $M' \lesssim 10$ MeV for the direct and inverse neutrino mass hierarchies.

As shown in Section 8, the lightest meV neutrino remains the only neutrino contribution to dark energy. The heavier neutrinos dissipate at the time of the age of the universe.

As shown in Section 7, the meV axion lifetime to decay into photons is much longer than the age of the universe. Dissipation of the energy in the cosmological quantum axion vacuum takes longer than the age of the universe too.

These results are unified in Section 9, with both light meV particles, meV axions and meV light neutrinos, contributing to dark energy together. Table 1 summarizes their contributions together with the computed equation of state.

On the other hand, let us mention that a global analysis of cosmological constraints on decaying axion-like particles (ALPs) performed recently ref. [67] shows that ALPs are stable on cosmological time scales unless they are heavy enough, with masses >300 keV. This is an independent confirmation that 10^{-3} eV axions, as shown in this paper, are safely stable enough to be considered as the source of dark energy. Previously, ALPs have been proposed, among other proposals, to be constituents of the cosmological energy density, i.e., ref. [68].

Table 1. The Equation of state today, w(0) + 1, computed from Equation (80) in three relevant cases, which all describe the dark energy observed today (Equation (89)): (i) no neutrino contribution to the dark energy; (ii) a Majorana neutrino contribution with mass $m = 3.2$ meV; (iii) a Dirac neutrino contribution with mass $m = 3.2$ meV. See the discussion in Section 9.

Neutrino Type	Scalar Mass	Equation of State Today
No vacuum neutrino energy	$M = 3.96$ meV	$w(0) + 1 = -0.00794$
Majorana neutrino $m = 3.2$ meV	$M = 4.35$ meV	$w(0) + 1 = -0.00473$
Dirac neutrino $m = 3.2$ meV	$M = 4.66$ meV	$w(0) + 1 = -0.00156$

In conformal time (η), the scalar and fermion fields rescaled by the scale factor $a(\eta)$ turn out to obey equations of motion similar to those in Minkowski space-time but with time-dependent masses

$$\chi'' - \nabla^2\chi + \left[M^2 a^2(\eta) - \frac{a''(\eta)}{a(\eta)} \right]\chi(\vec{x},\eta) = 0,$$
$$[\, i \not\partial - m\, a(\eta)\,]\, \psi(\vec{x},\eta) = 0. \tag{3}$$

Here, χ and ψ are, respectively, rescaled scalar and fermion fields, ∇^2 is the usual flat space Laplacian and $i\not\partial$ is the usual Dirac differential operator in Minkowski space-time in terms of flat space-time Dirac matrices.

There are two widely separate scales in the field evolution in cosmological space-times:

- The fast scale is the microscopic quantum evolution scale, typically$\sim 1/M \sim 1/m$, where M and m are the scalar and fermion masses, respectively.
- The slow scale is the Hubble scale $1/H$ of the universe expansion. When $M \sim m \gg H$, $M^2 \gg a''(\eta)/a^3(\eta)$, and hence the scale factor can be considered as constant.
- Therefore, the cosmological quantum field evolution for the fields χ and ψ is just the Minkowski evolution with effective masses $(M^2\, a^2)$ and $(m\, a)$, respectively, as seen from Equation (3).

Energy density, pressure and field density are expressed in field theory as products of the field operators and their derivatives at equal space-time points. Such expressions are ultraviolet divergent and need to be subtracted. The subtractions respect the symmetries of the theory, and we normalize them such that the physical quantities are zero for the vacuum in Minkowski space-time. The finite resulting quantities grow as $\log a(\eta)$. This is analogous to the high-energy growth of renormalized one-loop Feynman graphs.

That is, the energy density and the pressure receive contributions from the quantum vacuum that grow as $\log a(\eta)$, while the particle contributions are as dilute as $1/a^3(\eta)$, as it must be for massive particles.

We obtain for the vacuum energy density and pressure of scalar and fermion fields with mass M and m, respectively, the following results:

$$< \mathcal{H} > (\eta) \overset{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} \frac{M^4}{2\,(4\,\pi)^2}\left[\log a(\eta) + b_S - \frac{1}{4} \right] - \frac{m^4}{(4\,\pi)^2}\,\mathcal{N}\left[\log a(\eta) + b_F - \frac{1}{4} \right], \tag{4}$$

$$< P > (\eta) \overset{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} -\frac{M^4}{2\,(4\,\pi)^2}\left[\log a(\eta) + b_S + \frac{1}{12} \right] + \frac{m^4}{(4\,\pi)^2}\,\mathcal{N}\left[\log a(\eta) + b_F + \frac{1}{12} \right] \tag{5}$$

where b_S and b_F take into account the initial values of the scale factor $a_{\rm dcs}$ and $a_{\rm dcf}$ (at the decoupling time) of the scalars and fermions, respectively. $\mathcal{N} = 1$ for Majorana fermions and $\mathcal{N} = 2$ for Dirac fermions.

Therefore, we obtain for the equation of state the explicit expression:

$$w(\eta) \;\equiv\; \frac{< P > (\eta)}{< \mathcal{H} > (\eta)} \stackrel{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} -1 - \frac{1}{3}\left[\log a(\eta) - \frac{1}{4} + \frac{b_S - (2\,\mathcal{N}\, m^4/M^4)\, b_F}{1 - (2\,\mathcal{N}\, m^4/M^4)}\right]^{-1}. \tag{6}$$

That is, we find $w(\eta) < -1$ with $w(\eta)$ asymptotically reaching the value -1 from below.

It is convenient to express the scale factor in terms of the redshift. Taking into account that b_S and b_F contain the initial values of the scale factor yields

$$a(\eta)\, e^{b_S} \;=\; \frac{1\,+\,z_S}{1+z}, \qquad a(\eta)\, e^{b_F} \;=\; \frac{1\,+\,z_F}{1+z}, \tag{7}$$

where z_S (z_F) is the redshift when the scalar (fermion) field is decoupled. For neutrinos, $z_F \sim 6 \times 10^9$, while, for axions with mass$\sim$1 meV, $z_S \sim 2.2 \times 10^{18}$.

We find from Equations (4) and (7),

$$< \mathcal{H} > (z) = \frac{1}{2\,(4\,\pi)^2}\left\{M^4\;\log z_S - 2\,\mathcal{N}\, m^4\;\log z_F - (M^4 - 2\,\mathcal{N}\, m^4)\left[\log(1+z) + \frac{1}{4}\right]\right\} \tag{8}$$

$$< P > (z) = -\frac{1}{2\,(4\,\pi)^2}\left\{M^4\;\log z_S - 2\,\mathcal{N}\, m^4\;\log z_F - (M^4 - 2\,\mathcal{N}\, m^4)\left[\log(1+z) - \frac{1}{12}\right]\right\},$$

where we used the conditions $z_S \gg 1,\; z_F \gg 1$.

We identify the vacuum energy density today $< \mathcal{H} > (z = 0)$ with the observed dark energy ρ_Λ. We can then write Equations (4), (6) and (8) as:

$$\rho_\Lambda \;=\; \frac{1}{2\,(4\,\pi)^2}\left[\,M^4\left(\log z_S - \frac{1}{4}\right) - 2\,\mathcal{N}\, m^4\left(\log z_F - \frac{1}{4}\right)\right], \tag{9}$$

$$< \mathcal{H} > (\eta) \stackrel{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} \rho_\Lambda\left[\,1\,+\,\beta_{\mathcal{N}}\;\log\frac{a(\eta)}{a_0}\,\right],$$

$$w(\eta) +;\, 1 \stackrel{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} -\;\frac{(M^4\;-\;2\;\mathcal{N}\; m^4)}{6\,(4\,\pi)^2\,\rho_\Lambda\left[\,1\,+\,\beta_{\mathcal{N}}\;\log\frac{a(\eta)}{a_0}\,\right]}\;, \tag{10}$$

where a_0 is the scale factor today and

$$\beta_{\mathcal{N}} \;=\; \frac{\left(1 - \dfrac{2\,\mathcal{N}\, m^4}{M^4}\right)}{\log z_S\;-\;\dfrac{1}{4}\;-\;\left(2\,\dfrac{\mathcal{N}\, m^4}{M^4}\right)\left[\,\log z_F - \dfrac{1}{4}\,\right]}. \tag{11}$$

That is, the vacuum energy density at late times after decoupling grows as the logarithm of the scale factor and the equation of state asymptotically approaches -1 from below.

The equation of state as a function of z takes the form:

$$w(z)\;+\;1\;=\;-\;\frac{1}{3}\;\frac{\left(1 - \dfrac{2\,\mathcal{N}\, m^4}{M^4}\right)}{\log z_S - \left(\dfrac{2\,\mathcal{N}\, m^4}{M^4}\right)\log z_F - \left(1 - \dfrac{2\,\mathcal{N}\, m^4}{M^4}\right)\left[\,\log(1+z) + \dfrac{1}{4}\,\right]}. \tag{12}$$

For $z = 0$, it becomes, today:

$$w(0) + 1 = -\frac{1}{3}\,\frac{\left(1 - \dfrac{2\,\mathcal{N}\,m^4}{M^4}\right)}{\log z_S - \dfrac{1}{4} - \left(\dfrac{2\,\mathcal{N}\,m^4}{M^4}\right)\left[\log z_F - \dfrac{1}{4}\right]} = -\frac{1}{6\,(4\,\pi)^2\,\rho_\Lambda}\left(M^4 - 2\,\mathcal{N}\,m^4\right) . \tag{13}$$

The scalar and fermion masses are constrained by the value of the dark energy today Equation (2). This gives the positivity requirement:

$$M > (2\,\mathcal{N})^{\frac{1}{4}}\,m,$$

as well as the expression for the mass of the scalar particle:

$$M = \frac{10.1 \;\text{meV}}{\left(\log z_S - \frac{1}{4}\right)^{\frac{1}{4}}}\left[1 + \mathcal{N}\left(\frac{m}{3.90 \;\text{meV}}\right)^4\right]^{\frac{1}{4}} . \tag{14}$$

The neutrino contribution to dark energy can be ignored when $m \ll 1$ meV and when the vacuum neutrino contribution dissipates in the time scale of the age of the universe, as mentioned before. The mass of the lightest neutrino is not yet known (only neutrino mass differences are known). We will consider that the lightest neutrino mass is either $m = 3.2$ meV [69,70] or zero [71,72].

More specifically, we set $z_S \sim 2.2 \times 10^{18}$, assuming the scalar field to be an axion with mass$\sim$1 meV in Equations (13) and (14).

- We therefore obtain for the axion mass M and for the equation of state today the following values:

$$3.96 \;\text{meV} < M < 4.66 \;\text{meV},$$
$$-0.00794 < w(0) + 1 < -0.00156 . \tag{15}$$

  The left and right ends of the intervals in Equation (15) correspond to no neutrino contribution and to the lightest neutrino contribution, respectively, as a Dirac fermion with mass $m = 3.2$ meV.
- We see that $w(0)$ is slightly below -1 by an amount ranging from (-1.5×10^{-3}) to (-8×10^{-3}), while the axion mass results are between 4 and 5 meV, which is within the range of axion masses allowed by astrophysical and cosmological constraints, e.g., [73–75].
  If the scalar particle is not the axion, the value of $z_S \gg 1$ will depend on the dynamics of such scalar particle.
- In general, we express the contribution of the quantum vacuum of light particles to dark energy and pressure in terms of two parameters: the particle masses and the redshifts when they are decoupled. There is also a dependence on the number of states per particle (1 for a scalar, $2\,\mathcal{N}$ for a fermion).
- We uncover in this paper the general mechanism producing the dark energy today. This mechanism is only based on well-known quantum physics and cosmology. The observed dark energy in the universe today appears as a quantum vacuum effect only due to the (classical) cosmological space-time expansion. That is to say, dark energy in the present universe is a semiclassical gravity effect.
- The dark energy arises for a quantum field in the cosmological context in an analogous way to how the Casimir effect arises for a quantum field in Minkowski space-time with non-trivial boundary conditions in space.
- All physical (finite) results are independent of any energy cutoff as well as of the regularization method used.

- We obtain and solve in this paper the self-consistent Einstein–Friedmann equation for the scale factor when dark energy dominates and the universe expansion accelerates. The growth of the energy density Equation (4) as the logarithm of the scale factor implies an expansion faster than in de Sitter space-time. More precisely, we find that the Universe will reach in the future an asymptotic phase where it expands exponentially as

$$a(t) \overset{H_0\, t \gtrsim 1}{\simeq} a\,(\text{today})\ \exp\left[\, c_1\, H_0\, t \,+\, c_2\ (H_0\ t)^2\, \right], \tag{16}$$

where

$$c_1 \;\equiv\; \sqrt{\Omega_\Lambda} \;=\; 0.87, \quad 0.00452 \;<\; c_2 \;<\; 0.00872, \tag{17}$$

and H_0 stands for the Hubble parameter today. The left and right ends of the interval for c_2 in Equation (17) correspond to no neutrino contribution and to the lightest neutrino contribution, respectively, as a Dirac fermion with mass $m = 3.2$ meV.
- Notice that the time scale of the accelerated expansion is huge:$\sim 1\,/\,H_0 = 13.4$ Gyr. In the exponent of Equation (16), the quadratic term dominates over the linear term by a time $t \sim 100\,/\,H_0$ to $200\,/\,H_0$.
  In this accelerated universe, we see from the Friedman equation and Equation (4) that the Hubble radius $1/H$ decreases with time as $1\,/\,[\,H_0\,\sqrt{\log a(t)}\,]$.

This paper is organized as follows: In Sections 2 and 3, we review the dynamics of scalar and fermion fields on cosmological space-times, respectively. In Section 4, we discus the quantum cosmological vacuum and the two point functions and compute the main physical quantities from them. In Section 5, we find the vacuum energy density and pressure and the equation of state for late times, and in Section 6 we discuss their quantum nature. In Section 7, we find dark energy as a result of the cosmological quantum vacuum contributions from meV light particles, and the properties, masses and stabilities of axions are treated.. In Section 8, we compute and analyze the neutrino contributions to dark energy: the lightest neutrino remains the only contribution. In Section 9, we unify these results with both light meV particles together. We obtain the future self-consistent evolution of the universe in Section 10. We discuss relevant related issues in Section 11, and we present our conclusions in Section 12. Appendix A is devoted to the equivalence between different regularization methods.

2. Scalar Fields in Cosmological Space-Times

We consider a massive neutral scalar field φ in an FRW geometry defined by the invariant distance

$$ds^2 \;=\; dt^2 \;-\; a^2(t)\; d\vec{x}^2. \tag{18}$$

The Lagrangian density is taken to be

$$\mathcal{L} \;=\; \frac{1}{2}\,\sqrt{-g}\,\left[\,\dot{\varphi}^2 \;-\; \left(\frac{\vec{\nabla}\varphi}{a}\right)^2 \;-\; M^2\,\varphi^2\,\right]. \tag{19}$$

It is convenient to use the conformal time η,

$$\eta \;=\; \int \frac{dt}{a(t)},$$

and the conformally rescaled field $\chi(\vec{x},\eta)$,

$$\chi(\vec{x},\eta) \;\equiv\; a(t)\;\varphi(\vec{x},t). \tag{20}$$

The action (after discarding surface terms that do not affect the equations of motion) reads:

$$A\Big(\chi,\delta\Big) \;=\; \frac{1}{2}\int d^3x\, d\eta \left[\chi'^2 - (\nabla\chi)^2 - \mathcal{M}^2(\eta)\,\chi^2 \right], \tag{21}$$

where primes denote derivatives with respect to the conformal time η and where

$$\mathcal{M}^2(\eta) \;=\; M^2\, a^2(\eta) \;-\; \frac{a''(\eta)}{a(\eta)} \tag{22}$$

plays the role of an effective mass squared. Therefore, the rescaled field $\chi(\vec{x},\eta)$ obeys the equation of motion,

$$\chi'' \;-\; \nabla^2\chi \;+\; \mathcal{M}^2(\eta)\,\chi \;=\; 0. \tag{23}$$

The evolution of $\chi(\vec{x},\eta)$ is similar to that of a scalar field in Minkowski space-time with a time-dependent mass squared $\mathcal{M}^2(\eta)$.

The solution for the field $\varphi(\vec{x},t)$ can be Fourier expanded as follows,

$$\varphi(\vec{x},\eta) \;=\; \frac{1}{a(\eta)}\int \frac{d^3k}{(2\,\pi)^3\, 2\, E_0}\left[a_{\vec{k}}\;\; \phi_k\,(\eta)\, e^{i\vec{k}\cdot\vec{x}} \;+\; a^{\dagger}_{\vec{k}}\;\; \phi^*_k\,(\eta)\, e^{-i\vec{k}\cdot\vec{x}} \right], \tag{24}$$

where

$$E_0 \;\equiv\; \sqrt{k^2 \;+\; \mathcal{M}_i^2}$$

and $\mathcal{M}_i$ is the effective mass $\mathcal{M}(\eta)$ at the decoupling time (initial time) for the scalar field evolution. The mode functions $\phi_k(\eta)$ obey the evolution equations,

$$\left[\frac{d^2}{d\eta^2} \;+\; k^2 \;+\; M^2\, a^2(\eta) \;-\; \frac{a''(\eta)}{a(\eta)} \right]\phi_k(\eta) \;=\; 0. \tag{25}$$

We choose the initial state as the vacuum state, which, here (at decoupling), is a thermal equilibrium state at temperature T. However, as we see below (Equation (70)), the effect of the initial temperature on the vacuum energy is negligible for late times. The Fock vacuum state $|0>$ is annihilated by the operators $a_{\vec{k}}$. Therefore, we have as initial conditions for the mode functions

$$\phi_k(0) \;=\; 1, \quad \phi'_k(0) \;=\; -\,i\,E_0. \tag{26}$$

These initial conditions describe the Bunch–Davies vacuum when they are applied at asymptotically earlier times in the past ($\eta \to -\infty$) [76,77]. See the discussion in Section 11 below.

The time-dependent creation and annihilation operators obey the canonical commutation rules,

$$\left[a_{\vec{k}}\;,\; a^{\dagger}_{\vec{k}'} \right] \;=\; 2\,E_0\,(2\,\pi)^3\;\delta(\vec{k}-\vec{k}').$$

The energy-momentum tensor for a scalar field is given by [76],

$$T_{\mu\nu} \;=\; \partial_\mu\varphi\,\partial_\nu\varphi \;-\; \frac{1}{2}\,g_{\mu\nu}\Big[\,\partial_\lambda\varphi\,\partial^\lambda\varphi \;-\; M^2\,\varphi^2\,\Big]. \tag{27}$$

Its expectation value has the fluid form

$$< T^0_{S\,0} > \;=\; < \mathcal{H}_S > (\eta), \quad < T^j_i > \;=\; -\;\delta^j_i < P_S > (\eta)$$

since we consider homogeneous and isotropic quantum states and density matrices. In conformal time, the hamiltonian density and the pressure take the form

$$\mathcal{H}_S(\eta) \;=\; \frac{1}{2\,a^4(\eta)}\Big\{\;[\,\chi'(\vec{x},\eta) - a(\eta)\;H(\eta)\;\chi(\vec{x},\eta)\,]^2 + (\,\nabla\chi(\vec{x},\eta)\,)^2 + a^2(\eta)\;M^2\;\chi^2(\vec{x},\eta)\;\Big\},$$

$$\mathcal{H}_S \,+\, P_S(\eta) \;=\; \frac{1}{a^4(\eta)}\left\{ [\, \chi'(\vec{x},\eta) - a(\eta)\; H(\eta)\; \chi(\vec{x},\eta) \,]^2 + \frac{1}{3}\; (\, \nabla\chi(\vec{x},\eta)\,)^2 \right\}, \tag{28}$$

where $H(\eta)$ stands for the Hubble parameter

$$H(\eta) \;\equiv\; \frac{d\; \ln a(t)}{dt} \;=\; \frac{1}{a^2(\eta)}\frac{da}{d\eta}. \tag{29}$$

It is convenient to consider the conformal energy and pressure,

$$\varepsilon_S(\eta) \;\equiv\; a^4(\eta)\;\; < \mathcal{H}_S > (\eta), \quad p_S(\eta) \;\equiv\; a^4(\eta)\;\; < P_S > (\eta). \tag{30}$$

We find the trace of the energy-momentum tensor from Equations (28),

$$a^4(\eta)[\, \mathcal{H}_S(\eta) - 3\; P_S(\eta)\,] \;=\; a^2(\eta)\; M^2\; \chi^2 - \left[(\chi' - a\; h\; \chi)^2 - (\nabla\chi)^2 - a^2(\eta)\; M^2\; \chi^2 \right]. \tag{31}$$

Ignoring the bracket term on the right hand side yields the virial theorem. Although this bracket term is non-zero, its space and time average is zero:

$$\frac{1}{\Delta}\int_\eta^{\eta+\Delta} d\eta \int d^3x \left[(\chi' - a\; h\; \chi)^2 - (\nabla\chi)^2 - a^2(\eta)\; M^2\; \chi^2 \right] \overset{\Delta \gg 1/M}{=} 0.$$

In addition, this bracket can be neglected for late times, as we shall see below.

Therefore, we have for the expectation values

$$\varepsilon_S(\eta) - 3\, p_S(\eta) \;=\; M^2\; a^2(\eta)\; \Sigma_S(\eta) - a^4(\eta)\; V(\eta), \tag{32}$$

where

$$\Sigma_S(\eta) \;\equiv < \chi^2(\vec{x},\eta) > = \; a^2(\eta)\;\; < \varphi^2(\vec{x},\eta) > \tag{33}$$

and V stands for the expectation value of the virial

$$V(\eta) \;\equiv < (\chi' - a\; h\; \chi)^2 - (\nabla\chi)^2 - a^2(\eta)\; M^2\; \chi^2 > .$$

Using the equations of motion (23), we obtain for the time derivative of the energy density Equation (30),

$$\frac{d\varepsilon_S}{d\eta} \;=\; \frac{1}{2}\; M^2\; \frac{da^2(\eta)}{d\eta}\;\; \Sigma_S(\eta) \;-\; a(\eta)\; H(\eta)\; V(\eta). \tag{34}$$

This relation in conformal time implies the usual continuity equation in cosmic time:

$$\frac{d}{dt} < \mathcal{H}_S > \,+\, 3\; H(\eta)\; [< \mathcal{H}_S > \,+\, < P_S >] \;=\; 0. \tag{35}$$

Therefore, from Equations (32) and (34), we see that there is only one independent quantity among $\varepsilon_S(\eta)$, $p_S(\eta)$ and $\Sigma_S(\eta)$.

3. Fermion Fields in Cosmological Space-Times

The Lagrangian density for fermions is taken to be [77]

$$\mathcal{L} \;=\; \sqrt{-g}\;\; \overline{\Psi} \left[\, i\, \gamma^\mu\; \mathcal{D}_\mu \Psi \;-\; m \,\right] \Psi. \tag{36}$$

The γ^μ are the curved space-time Dirac γ matrices, and the fermionic covariant derivative is given by

$$\mathcal{D}_\mu \;\;=\;\; \partial_\mu \,+\, \frac{1}{8}\, [\, \gamma^c \,,\, \gamma^d_{;}]\; V^\nu_c\; (\, D_\mu V_{d\nu}\,)$$

$$D_\mu V_{d\nu} \;=\; \partial_\mu V_{d\nu} \;-\; \Gamma^\lambda_{\mu\nu}\, V_{d\lambda},$$

where the vierbein field is defined as

$$g^{\mu\,\nu} \;=\; V^\mu_a\, V^\nu_b\, \eta^{ab},$$

η_{ab} is the Minkowski space-time metric and the curved space-time matrices γ^μ are given in terms of the Minkowski space-time ones, γ^a (Greek indices refer to curved space-time coordinates and Latin indices to the local Minkowski space-time coordinates):

$$\gamma^\mu \;=\; \gamma^a\, V^\mu_a, \quad \{\gamma^\mu, \gamma^\nu\} \;=\; 2\, g^{\mu\nu}.$$

In conformal time, the vierbeins V^μ_a are particularly simple:

$$V^\mu_a \;=\; a(\eta)\, \delta^\mu_a, \tag{37}$$

where $a(\eta) \equiv a(t(\eta))$ is the scale factor as a function of the conformal time and we call $a(\eta = 0) = a_{\rm dc}$. The Dirac Lagrangian density thus simplifies to the following expression:

$$\sqrt{-g}\;\overline{\Psi}\big(\, i\,\gamma^\mu\, \mathcal{D}_\mu \Psi \;-\; m\,\big)\Psi \;=\; a^{\frac{3}{2}}\;\overline{\Psi}\;[\; i\, \not{\partial} \;-\; m\, a(\eta)\;]\Big(a^{\frac{3}{2}}\Psi\Big), \tag{38}$$

where $i\,\not{\partial}$ is the usual Dirac differential operator in Minkowski space-time in terms of flat space-time γ^a matrices.

Therefore, the Dirac equation in the FRW geometry is given by

$$[\; i\, \not{\partial} \;-\; m\, a(\eta)\;]\Big[\, a^{\frac{3}{2}}\; \Psi(\vec{x}, \eta)\,\Big] \;=\; 0. \tag{39}$$

The solution $\Psi(\vec{x}, \eta)$ can be expanded in spinor mode functions as

$$\Psi(\vec{x}, \eta) \;=\; \frac{1}{a^{\frac{3}{2}}(\eta)} \sum_{\lambda=\pm 1} \int \frac{d^3k}{(2\,\pi)^3\; 2\; e_0}\; e^{i\vec{k}\cdot\vec{x}}\, \Big[\, b_{\vec{k},\lambda}\, U_\lambda(\vec{k}, \eta) \;+\; d^\dagger_{-\vec{k},\lambda}\, V_\lambda(-\vec{k}, \eta)\,\Big], \tag{40}$$

where

$$e_0 \;\equiv\; \sqrt{k^2 \;+\; m^2\, a^2_{\rm dc}}$$

and the spinor mode functions U, V obey the Dirac equations

$$\Big[\, i\,\gamma^0\, \partial_\eta \;-\; \vec{\gamma}\cdot\vec{k} \;-\; m\, a(\eta)\,\Big] U_\lambda(\vec{k}, \eta) \;=\; 0, \tag{41}$$

$$\Big[\, i\,\gamma^0\, \partial_\eta \;+\; \vec{\gamma}\cdot\vec{k} \;-\; m\, a(\eta)\,\Big] V_\lambda(\vec{k}, \eta) \;=\; 0. \tag{42}$$

The time-independent creation and annihilation operators obey the canonical anticommutation rules

$$\Big\{ b_{\vec{k},\lambda}\,,\; b^\dagger_{\vec{k}',\lambda'} \Big\} \;=\; 2\, e_0\; (2\,\pi)^3\; \delta(\vec{k} - \vec{k}')\; \delta_{\lambda\,\lambda'},$$

$$\Big\{ d_{\vec{k},\lambda}\,,\; d^\dagger_{\vec{k}',\lambda'} \Big\} \;=\; 2\, e_0\; (2\,\pi)^3\; \delta(\vec{k} - \vec{k}')\; \delta_{\lambda\,\lambda'}. \tag{43}$$

Following the method of refs. [41,42], it proves convenient to write

$$U_\lambda(\vec{k}, \eta) \;=\; (e_0 \;+\; m\, a_{\rm dc})^{-\frac{1}{2}}\, \Big[\, i\,\gamma^0\, \partial_\eta \;-\; \vec{\gamma}\cdot\vec{k} \;+\; m\, a(\eta)\,\Big] f_k(\eta)\; \mathcal{U}_\lambda \tag{44}$$

$$V_\lambda(\vec{k}, \eta) \;=\; (e_0 \;+\; m\, a_{\rm dc})^{-\frac{1}{2}}\, \Big[\, i\,\gamma^0\, \partial_\eta \;+\; \vec{\gamma}\cdot\vec{k} \;+\; m\, a(\eta)\,\Big] g_k(\eta)\; \mathcal{V}_\lambda, \tag{45}$$

with $(e_0 + m \, a_{\rm dc})^{-\frac12}$ being a normalization factor and $(\mathcal{U}_\lambda, \; \mathcal{V}_\lambda)$ being constant spinors [41,42] obeying

$$\gamma^0 \, \mathcal{U}_\lambda \; = \; \mathcal{U}_\lambda \quad , \quad \gamma^0 \, \mathcal{V}_\lambda \; = \; - \, \mathcal{V}_\lambda \quad , \quad \lambda \; = \; \pm \, 1. \tag{46}$$

More explicitly,

$$U_\lambda(\vec{k},\eta) \; = \; (e_0 \; + \; m \; a_{\rm dc})^{-\frac12} \begin{pmatrix} [\, i \, f'_k(\eta) \; + \; m \, a(\eta) \, f_k(\eta) \,] & 0 \\ 0 & \lambda \, k \, f_k(\eta) \end{pmatrix} \mathcal{U}_\lambda \, ,$$

$$V_\lambda(-\vec{k},\eta) = (e_0 \; + \; m \; a_{\rm dc})^{-\frac12} \begin{pmatrix} \lambda \, k \, g_k(\eta) & 0 \\ 0 & [\, -i \, g'_k(\eta) \; + \; m \, a(\eta) \, g_k(\eta) \,] \end{pmatrix} \mathcal{V}_\lambda \, . \tag{47}$$

The mode functions $f_k(\eta), \; g_k(\eta)$ obey then the following equations of motion

$$\left[\frac{d^2}{d\eta^2} \; + \; k^2 \; + \; m^2 \, a^2(\eta) \; - i \, m \, a'(\eta) \right] f_k(\eta) \quad = \quad 0 \tag{48}$$

$$\left[\frac{d^2}{d\eta^2} \; + \; k^2 \; + \; m^2 \, a^2(\eta) \; + \; i \, m \, a'(\eta) \right] g_k(\eta) \quad = \quad 0. \tag{49}$$

We choose the initial state for the fermion field as the vacuum state, which is a thermal equilibrium state at temperature T for the fermion. This Fock state $|0>$ is annihilated by the operators $b_{\vec{k},\lambda}$ and $d_{\vec{k},\lambda}$.

Therefore, we have as initial conditions for the mode functions [41,42]

$$f_k(0) \; = \; 1, \quad f'_k(0) \; = \; - \, i \, e_0, \tag{50}$$

$$g_k(0) \; = \; 1, \quad g'_k(0) \; = \; + \, i \, e_0.$$

These initial conditions describe the Bunch–Davies vacuum when they are applied at asymptotically earlier times in the past ($\eta \to -\infty$) [76,77]. See the discussion in Section 11 below.

Equations (48)–(50) imply that

$$g_k(\eta) \; = \; f^*_k(\eta).$$

That is, we have only one independent and complex mode function.

The scalar products of the spinors $U_\lambda(\vec{k},\eta), \; V_\lambda(\vec{k},\eta)$ take the values

$$U^\dagger_\lambda \, (\vec{k},\eta) \, U_{\lambda'}(\vec{k},\eta) \;\; = \;\; 2 \, e_0 \, \delta_{\lambda \, \lambda'}$$

$$V^\dagger_\lambda \, (\vec{k},\eta) \, V_{\lambda'}(\vec{k},\eta) \;\; = \;\; 2 \, e_0 \, \delta_{\lambda \, \lambda'}. \tag{51}$$

As a consequence, the mode functions obey the relation [41,42]

$$|f'_k(\eta)|^2 - i \, m \, a(\eta) \, [f_k(\eta) \, f'^*_k(\eta) - f'_k(\eta) \, f^*_k(\eta)] + [k^2 + m^2 \, a^2(\eta)] \, |f_k(\eta)|^2 = 2 \, e_0(e_0 + m \, a_{\rm dc}),$$

which provides a conserved quantity.

The energy momentum tensor for a spin $1/2$ field is given by [76]

$$T^F_{\mu\nu} \;\; = \;\; \frac{i}{2} \left[\overline{\Psi} \gamma_{(\mu} \overleftrightarrow{\mathcal{D}}_{\nu)} \, \Psi \right], \tag{52}$$

and its expectation value has the fluid form

$$< T_{F0}^{\;\;0} > \; = \; < \mathcal{H}_F > (\eta), \quad < T_{F i}^{\;\;j} > \; = \; - \, \delta^j_i \; < P_F > (\eta)$$

since we consider homogeneous and isotropic quantum states and density matrices.

More explicitly, the energy density in conformal time takes the form

$$< \mathcal{H}_F > (\eta) \;=\; < \Psi(\vec{x},\eta)^\dagger \, H_F \, \Psi(\vec{x},\eta) >, \tag{53}$$

where the fermion hamiltonian H_F is defined by

$$a(\eta)\,\gamma_0\,H_F \;=\; -\,i\vec{\gamma}\cdot\vec{\nabla} \;+\; m\,a(\eta) \;=\; \vec{\gamma}\cdot\vec{p} \;+\; m\,a(\eta). \tag{54}$$

An analogous expression can be written for the pressure,

$$< P_F > (\eta) \;=\; \frac{1}{3\,a(\eta)} \; < \bar{\Psi}\,\vec{\gamma}\cdot\vec{p}\,\Psi > (\eta). \tag{55}$$

Here, too, it is convenient to consider the conformal energy and pressure,

$$\varepsilon_F(\eta) \;\equiv\; a^4(\eta) \; < \mathcal{H}_F > (\eta), \quad p_F(\eta) \;\equiv\; a^4(\eta) \; < P_F > (\eta). \tag{56}$$

We find the trace of the energy-momentum tensor from Equations (54)–(56),

$$\varepsilon_F(\eta) \;-\; 3\,p_F(\eta) \;=\; m\,a(\eta)\,\Sigma_F(\eta), \qquad \text{or} \tag{57}$$

$$< \mathcal{H}_F > (\eta) \;-\; 3\; < P_F > (\eta) \;=\; m \; < \bar{\Psi}\Psi > (\eta).$$

This is the expression of the virial theorem in the present context and

$$\Sigma_F(\eta) \;\equiv\; a^3(\eta) \; < \bar{\Psi}\Psi > (\eta). \tag{58}$$

The above expressions for the energy density and pressure obey the usual continuity equation in cosmic time:

$$\frac{d}{dt} < \mathcal{H}_F > \; + 3\,H(\eta)\,(< \mathcal{H}_F > + < P_F >) = 0, \tag{59}$$

In conformal time, by using Equations (57) and (58), the continuity Equation (59) becomes

$$\frac{d\varepsilon_F}{d\eta} \;=\; m\,\frac{da(\eta)}{d\eta}\,\Sigma_F(\eta). \tag{60}$$

We thus see from Equations (57) and (60) that there is only one independent quantity among $\varepsilon_F(\eta)$, $\mathcal{P}_F(\eta)$ and $\Sigma_F(\eta)$.

4. The Cosmological Quantum Vacuum

There are two widely separate scales in the field evolution in the cosmological space-time. The fast scale is the microscopic quantum evolution scale, typically$\sim 1/M \sim 1/m$. The slow scale is the Hubble scale, $1/H$ of the universe expansion.

When $M \sim m \gg H$, we can consider that the scale factor is practically constant. Therefore, in conformal time, the quantum field evolution is similar to the evolution in Minkowski space-time with a mass $M\,a(\eta)$ or $m\,a(\eta)$ for bosons or fermions, respectively (see Equations (22) and (39)).

The scalar and fermion densities follow as equal point limits of the scalar and fermion two-point functions. That is, we consider the scale factor a as a constant and obtain for the scalar two-point function

$$G_S(\vec{x}-\vec{x}',\eta-\eta',M\,a) \;\equiv\; < T\;\varphi(\vec{x},\eta)\;\varphi(\vec{x}',\eta') > = \frac{1}{a^2}\int \frac{d^4k}{(2\,\pi)^4}\,e^{-ik\cdot(x-y)}\,\frac{i}{k^2 - a^2\,M^2 + i\,0} =$$

$$= \frac{1}{(2\,\pi)^2}\,\frac{M}{z\,a}\,K_1(M\,a\,z) \quad , \quad z \equiv \sqrt{(\vec{x}-\vec{x}')^2-(\eta-\eta')^2}, \tag{61}$$

where $K_1(x)$ is a modified Bessel function.

Equation (61) is the zeroth order adiabatic approximation. It differs from the exact two-point function by quantities of the order $\mathcal{O}(a'(\eta)),\ \mathcal{O}(a''(\eta))$, etc.

We find from Equation (61) in the coincidence point limit:

$$G_S(\vec{x}-\vec{x}',\eta-\eta',M\,a) \overset{z\to 0}{=} \frac{1}{(2\,\pi)^2}\left\{\frac{1}{z^2\,a^2}+\frac{1}{2}\,M^2\left[\log(M\,a\,z)+\mathcal{C}-\ln 2-\frac{1}{2}\right]\right\}[\,1+\mathcal{O}(M^2\,z^2)\,], \tag{62}$$

where $\mathcal{C} = 0.57721566\ldots$ is the Euler–Mascheroni constant. Equations (61) and (62) display the two-point functions for the zero temperature vacuum. The effect of a non-zero temperature on the two-point function is negligible for $a \gg 1$, as we show below (Equation (70)).

The fermion two-point function takes the form

$$< T\,\Psi(\vec{x},\eta)_\alpha\,\bar{\Psi}(\vec{x}',\eta')_\beta > \;=\; \frac{1}{a^3}\int\frac{d^4k}{(2\,\pi)^4}\,e^{-ik\cdot(x-y)}\;\frac{i\,(\not{k}+a\,m)_{\alpha\,\beta}}{k^2-a^2\,m^2+i\,0}, \tag{63}$$

and hence,

$$G_F(\vec{x}-\vec{x}',\eta-\eta',m\,a) \equiv < T\,\bar{\Psi}(\vec{x},\eta)\,\Psi(\vec{x}',\eta') > = -\,4\,m\,G_S(\vec{x}-\vec{x}',\,\eta-\eta',\,m\,a), \qquad \text{Dirac fermions.} \tag{64}$$

The minus sign in front arose from the anticommutation of the fermion fields going from Equation (63) to Equation (64). Here, we used Equation (61) and

$$\mathrm{Tr}\,\not{k} \;=\; 0, \quad \mathrm{Tr}\,1 \;=\; 4. \tag{65}$$

That is, the factor $4 = 2\times 2$ in Equations (64) and (65) comes from the fermion and antifermion contributions times the number of helicities of a Dirac fermion. Hence, this factor 4 becomes a factor 2 for Majorana fermions:

$$G_F\,(\,\vec{x}-\vec{x}'\,,\,\eta-\eta'\,,\,m\,a\,) \;\equiv\; < \bar{\Psi}(\vec{x},\eta)\,\Psi(\vec{x}',\eta') > \;=\; -\,2\,m\,G_S\,(\vec{x}-\vec{x}'\,,\,\eta-\eta'\,,\,m\,a\,). \tag{66}$$

We find in the coincidence point limit corrections up to $[\,1+\mathcal{O}(m^2\,z^2)\,]$:

$$G_F(\vec{x}-\vec{x}',\eta-\eta',m\,a) \overset{z\to 0}{=} -\,\frac{2\,\mathcal{N}\,m}{(2\,\pi)^2}\left\{\frac{1}{z^2\,a^2}+\frac{1}{2}\,m^2\left[\log(m\,a\,z)+\mathcal{C}-\ln 2-\frac{1}{2}\right]\right\}. \tag{67}$$

Here, $\mathcal{N}=1$ for Majorana fermions and $\mathcal{N}=2$ for Dirac fermions.

In order to define the vacuum densities as the coincidence limits,

$$< \varphi^2 > (\eta) \;\equiv\; < \varphi^2(\vec{x},\eta) >, \quad < \bar{\Psi}\Psi > (\eta) \;\equiv\; < \bar{\Psi}(\vec{x},\eta)\,\Psi(\vec{x},\eta) >,$$

we have to subtract the singularities at $z=0$ in Equations (62) and (67). Subtracting the singularities leaves a finite z independent piece. Requiring that the vacuum densities vanish in Minkowski space-time ($a=1$), we obtain

$$< \varphi^2 > (\eta) = \frac{M^2}{2\,(2\,\pi)^2}\,[\,\log a + b_S\,f_S(a)\,], \quad < \bar{\Psi}\Psi > (\eta) = -\frac{\mathcal{N}\,m^3}{(2\,\pi)^2}\,[\,\log a + b_F\,f_F(a)\,]\,. \tag{68}$$

The functions $f_S(a)$ and $f_F(a)$ are finite and vanish for Minkowski space-time,

$$f_S(1) \;=\; 0, \quad f_F(1) \;=\; 0.$$

We compute the terms $b_S\ f_S(a)$ and $b_F\ f_F(a)$ with the result

$$f_S(\infty) \overset{a(\eta)\,\gg\, a_{\rm dcs},\, a_{\rm dcf}}{=} 1 + \mathcal{O}\left(\frac{1}{a^2}\right), \qquad f_F(\infty) \overset{a(\eta)\,\gg\, a_{\rm dcs},\, a_{\rm dcf}}{=} 1 + \mathcal{O}\left(\frac{1}{a^2}\right).$$

When one performs an infinite subtraction at $z = 0$, an additional finite subtraction can always be done. We recognize that the additional terms containing b_S and b_F can be absorbed in a finite multiplicative renormalization of the scale factor. That is, introducing b_S and b_F amounts to a scale transformation. We compute the coefficients b_S and b_F in terms of the subtraction scale in momentum space $(x\ M)$ for scalars and $(x\ m)$ for fermions, with the result

$$b_S(x) = b_F(x) = -\frac{1}{2} - \log x - \log a_{\rm dc},$$

where $a_{\rm dc}$ stands for the scale factor at decoupling time (initial time). In summary, we have for the late time regime,

$$< \varphi^2 > (\eta) \overset{a(\eta)\,\gg\, a_{\rm dcs},\, a_{\rm dcf}}{=} \frac{M^2}{2\,(2\,\pi)^2}\,[\log a(\eta) + b_S] = \frac{M^2}{2\,(2\,\pi)^2}\left[\log\frac{a(\eta)}{x\ a_{\rm dcs}} - \frac{1}{2}\right],$$
$$< \bar{\Psi}\Psi > (\eta) \overset{a(\eta)\,\gg\, a_{\rm dcs},\, a_{\rm dcf}}{=} -\frac{\mathcal{N}\ m^3}{(2\,\pi)^2}\,[\log a(\eta) + b_F] = -\frac{\mathcal{N}\ m^3}{(2\,\pi)^2}\left[\log\frac{a(\eta)}{x\ a_{\rm dcf}} - \frac{1}{2}\right], \tag{69}$$

where $a_{\rm dcs}$ and $a_{\rm dcf}$ stand for the scale factor at the decoupling times (initial times) for the scalar and the fermion, respectively.

The two-point function Equations (61) and (64) correspond to the zero-temperature case. The singular pieces for $z \to 0$ are temperature independent. We can disregard the temperature-dependent contributions to the two-point functions since for large a they decrease as

$$\sqrt{M}\left(\frac{T}{2\,\pi\,a}\right)^{\frac{3}{2}} e^{-\frac{M\,a}{T}} \to 0, \quad a \gg 1. \tag{70}$$

The scalar and fermion densities $< \varphi^2 > (\eta)$ and $< \bar{\Psi}\Psi > (\eta)$ can be also computed as momentum integrals over the mode functions $\phi_k(\eta)$ and $f_k(\eta)$. In addition, the subdominant terms in $1/a^2(\eta)$, $\dot{a}(\eta)/a^2(\eta)$, . . ., etc., can be obtained.

The equal points behavior of the two-point function Equations (62) and (67) is generic for any curved space-time when expressed as a function of the geodesic (squared) distance $\sigma \equiv z^2\ a^2$ between the two points. That is, the short distance behavior is uniquely and universally determined by the local space-time geometry. It must be noticed that the divergences and finite pieces at $\sigma = 0$ are of the same type as in Minkowski space-time. This is the so-called Hadamard expansion for $\sigma \to 0$ and is equivalent to the adiabatic expansion. The coefficients of the divergent and finite parts are called Hadamard coefficients and they are known for generic space-times.

5. Vacuum Energy Density and Pressure for Late Times

The total energy density $\varepsilon(\eta)$ and pressure $\mathcal{P}(\eta)$:

$$< \mathcal{H} > (\eta) = < \mathcal{H}_S > (\eta) + < \mathcal{H}_F > (\eta), \tag{71}$$

$$< P > (\eta) = < P_S > (\eta) + < P_F > (\eta), \tag{72}$$

can be computed in the late-time regime using the virial theorem Equations (32) and (57), the continuity equation Equations (34) and (60) and the late-time behavior of the densities, Equation (69).

We obtain after calculation for the energy density and pressure,

$$< \mathcal{H} > (\eta) \overset{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} \frac{M^4}{2\,(4\,\pi)^2}\left[\log a(\eta) + b_S - \frac{1}{4}\right] - \frac{m^4}{(4\,\pi)^2}\,\mathcal{N}\left[\log a(\eta) + b_F - \frac{1}{4}\right], \tag{73}$$

$$< P > (\eta) \overset{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} -\frac{M^4}{2\,(4\,\pi)^2}\left[\log a(\eta) + b_S + \frac{1}{12}\right] + \frac{m^4}{(4\,\pi)^2}\,\mathcal{N}\left[\log a(\eta) + b_F + \frac{1}{12}\right]. \tag{74}$$

The decoupling (initial) times for the evolution of scalars and fermions can be different from each other. We have absorbed in b_S and b_F the corresponding initial values of the scale factor for scalars and fermions, respectively.

The positivity of the energy density imposes the condition

$$M^4 \; > \; 2\,\mathcal{N}\,m^4.$$

Notice that

$$< P > (\eta) + \; < \mathcal{H} > (\eta) \overset{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} -\frac{1}{6\,(4\,\pi)^2}\left[M^4 \; - \; 2\,\mathcal{N}\,m^4\right]$$

is time independent and independent of the finite subtraction coefficients b_S and b_F as well.

From Equation (73), we obtain for the equation of state,

$$w(\eta) \;\equiv\; \frac{< P > (\eta)}{< \mathcal{H} > (\eta)} \overset{a(\eta) \gg a_{\rm dcs},\, a_{\rm dcf}}{=} -1 \;-\; \frac{1}{3}\left[\log a(\eta) \;-\; \frac{1}{4} \;+\; \frac{b_S - (\,2\,\mathcal{N}\,m^4/M^4\,)\,b_F}{1 - (\,2\,\mathcal{N}\,m^4/M^4\,)}\right]^{-1}. \tag{75}$$

That is, we find $w(\eta) < -1$ with $w(\eta)$ asymptotically reaching the value -1 from below.

It is convenient to express the scale factor in terms of the redshift as

$$a(\eta)\;\exp\left(b_S\right) \;=\; \frac{1\;+\;z_S}{1\;+\;z}, \qquad a(\eta)\;\exp\left(b_F\right) \;=\; \frac{1\;+\;z_F}{1\;+\;z}, \tag{76}$$

where z_S (z_F) is the redshift when the evolution of the scalar (fermion) becomes the one of a free field in cosmological space-time. In terms of z_S and z_F, Equation (73) reads,

$$\begin{aligned}
[\,2\,(4\,\pi)^2\,] \; < \mathcal{H} > (z) \;=\; & M^4 \log\left(1 + z_S\right) \;-\; 2\,\mathcal{N} m^4 \log\left(1 + z_F\right) \;- \\
& - \left(M^4 \;-\; 2\,\mathcal{N} m^4\right)\left[\log\left(1 + z\right) \;+\; \frac{1}{4}\right],
\end{aligned} \tag{77}$$

$$\begin{aligned}
[\,-\,2\,(4\,\pi)^2\,] \; < P > (z) \;=\; & M^4 \log\left(\,1 + z_S\,\right) \;-\; 2\,\mathcal{N}\,m^4 \log\left(\,1 \;+\; z_F\,\right) \;- \\
& - \left(\,M^4 \;-\; 2\,\mathcal{N}\,m^4\,\right)\left[\log\left(\,1 \;+\; z\,\right) \;-\; \frac{1}{12}\right].
\end{aligned} \tag{78}$$

The equation of state (75) as a function of z takes the form:

$$\begin{aligned}
w(z) \;+\; 1 \;=\; & -\tfrac{1}{3}\left(1 - \tfrac{2\,\mathcal{N}\,m^4}{M^4}\right)\times \\
\times \left\{\log\left(1 + z_S\right) - \left(\tfrac{2\,\mathcal{N}\,m^4}{M^4}\right)\log\left(1 + z_F\right) - \left(1 - \tfrac{2\,\mathcal{N}\,m^4}{M^4}\right)\left[\log\left(1 + z\right) + \tfrac{1}{4}\right]\right\}^{-1}&.
\end{aligned} \tag{79}$$

The equation of state and the energy density of today become:

$$w(z = 0) + 1 \;=\; -\frac{1}{3}\left(1 - \frac{2\,\mathcal{N}\,m^4}{M^4}\right)\left\{\log\left(1 + z_S\right) - \frac{1}{4} - \left(\frac{2\,\mathcal{N}\,m^4}{M^4}\right)\left[\log\left(1 + z_F\right) - \frac{1}{4}\right]\right\}^{-1}, \tag{80}$$

$$< \mathcal{H} > (z=0) \;=\; \frac{1}{2\,(4\,\pi)^2}\left\{M^4\left[\log\left(1+z_S\right)-\frac{1}{4}\right]-2\,\mathcal{N}\,m^4\left[\log\left(1+z_F\right)-\frac{1}{4}\right]\right\}. \tag{81}$$

The energy density at late times η after decoupling and the energy density today are related by

$$< \mathcal{H} > (\eta) \overset{a(\eta)\gg a_{\rm dcs},\, a_{\rm dcf}}{=} < \mathcal{H} > (z=0) \;+\; \left(\frac{M^4-2\,\mathcal{N}\,m^4}{2\,(4\,\pi)^2}\right)\,\log\!\left(\frac{a(\eta)}{a_0}\right), \tag{82}$$

where we used Equations (73) and (80) and a_0 stands for the scale factor today.

We identify the vacuum energy density today $< \mathcal{H} > (z=0)$ with the observed dark energy ρ_Λ. We can then write,

$$< \mathcal{H} > (\eta) \;=\; \rho_\Lambda\left[1+\beta_{\mathcal{N}}\;\log\frac{a(\eta)}{a_0}\right], \tag{83}$$

where

$$\beta_{\mathcal{N}} \;\equiv\; \left(1-\frac{2\,\mathcal{N}\,m^4}{M^4}\right)\left\{\log\left(1+z_S\right)\;-\;\frac{1}{4}\;-\;\left(\frac{2\,\mathcal{N}\,m^4}{M^4}\right)\left[\log\left(1+z_F\right)\;-\;\frac{1}{4}\right]\right\}^{-1}. \tag{84}$$

That is, the vacuum energy density at late times after decoupling grows as the logarithm of the scale factor. Moreover, the equation of state approaches -1 from below as:

$$w(\eta)+1 \overset{a(\eta)\gg a_{\rm dcs},\, a_{\rm dcf}}{=} -\left(\frac{M^4-2\,\mathcal{N}\,m^4}{6\,(4\,\pi)^2\,\rho_\Lambda}\right)\left[1\;+\;\beta_{\mathcal{N}}\;\log\frac{a(\eta)}{a_0}\right]^{-1}.$$

The previous equations in this subsection generalize when there are several scalar and fermion fields by just summing over their respective contributions. Let us consider the case of several scalars and fermions. This case is relevant to study whether the three neutrino mass eigenstates can contribute to dark energy. Equation (80) becomes for $z_S,\; z_F \gg 1$:

$$w(z\;=\;0)\;+\;1\;=\;-\;\frac{\sum_j M_j^4\;-\;2\,\mathcal{N}\;\sum_i m_i^4}{6\,(4\,\pi)^2\;\rho_\Lambda}, \tag{85}$$

$$\rho_\Lambda \;=\; \frac{1}{2\,(4\,\pi)^2}\left[\left(\log z_S-\frac{1}{4}\right)\sum_j M_j^4-2\,\mathcal{N}\left(\log z_F-\frac{1}{4}\right)\sum_i m_i^4\right], \tag{86}$$

where j and i label the species of scalars and fermions, respectively.

It is convenient to eliminate the sum of scalar masses $\sum_j M_j^4$ between Equations (85) and (86), with the result,

$$w(z\;=\;0)\;+\;1\;=\;\frac{1}{(\log z_S-\frac{1}{4})}\left[-\frac{1}{3}\;+\;\frac{\mathcal{N}}{3\,(4\,\pi)^2}\;\frac{\sum_i m_i^4}{\rho_\Lambda}\;\log\frac{z_S}{z_F}\right]. \tag{87}$$

We see in Equation (87) that the scalar contributes to the equation of state today by the negative term $-1/[\,3\,(\,\log z_S-\frac{1}{4}\,)\,]$, while the fermions give for $z_S > z_F$ a positive contribution proportional to the sum of the fourth power of their masses.

6. The Quantum Nature of the Cosmological Vacuum

Local observables as $< \varphi^2 >,\; < \bar{\Psi}\Psi >$, the energy density and the pressure involve the product of field operators at equal points. This is identical to one-loop tadpole Feynman diagrams. Logarithmic dependence on the scale of the momenta is typical in one-loop renormalized Feynman diagrams [78]. Here, we analogously find a logarithm of the scale factor in Equations (69) and (73) through the same mechanisms at work in renormalized quantum field theory. Hence, dark energy follows here as a truly quantum field vacuum effect. We stress quantum field effect and not just quantum effect because the infinite number of filled momentum modes in the vacuum as well as the subtraction of UV divergences play a crucial role in the vacuum late-time behavior. Here, the quantum fields are neither coupled nor self-coupled, but they interact with the expanding space-time geometry.

Notice that these results from Equations (69), (73) and (75) are valid for any expanding universe. They do not depend on the specific time dependence of the scale factor $a(\eta)$, provided it grows with η.

The quantum nature of the vacuum cosmological effects in the physical observables here are manifested from Equations (69) and (73),

$$\begin{aligned} < \varphi^2 > (\eta) &\sim M^2 \log a(\eta) = \frac{M^2\, c^2}{\hbar} \log a(\eta) = \frac{M\, c}{\lambda_C} \log a(\eta), \\ < \mathcal{H} > (\eta) &\sim M^4 \log a(\eta) = M\, c^2 \left(\frac{M\, c}{\hbar}\right)^3 \log a(\eta) = \frac{M\, c^2}{\lambda_C^3} \log a(\eta)\,. \end{aligned} \tag{88}$$

These quantities are of quantum nature since they depend on $\hbar$. There is no 'classical contribution' to the vacuum energy. Equation (88) just means that the scale of the dark energy density is of one scalar rest mass per a volume equal to the cube of the Compton wavelength λ_C for the scalar particle. Notice that $\lambda_C = [\hbar/(M\, c)] \simeq 0.05$ mm is almost a macroscopic length, while the mass of the scalar particle $M \sim 4$ meV $= 7.1\, 10^{-36}$ g is extremely small (see below for the value of M).

7. Dark Energy from the Cosmological Quantum Vacuum

Let us recall the current value for the dark energy density

$$\rho_\Lambda = \Omega_\Lambda\, \rho_c = 3.28 \times 10^{-11}\ (\text{eV})^4 = (2.39\ \ \text{meV})^4, \tag{89}$$

corresponding to

$$h = 0.73 \text{ and } \Omega_\Lambda = 0.76 \quad \text{and where } 1\ \text{meV} = 10^{-3}\ \text{eV}. \tag{90}$$

We take these values because they do correspond to direct, model-independent and late universe observations, refs. [1,4,5,8,13,79–82], and, accordingly, this paper deals directly with dark energy in the late universe; moreover, dark energy was discovered with such direct and model-independent measurements in the late universe, refs. [1,4,5,8]. Other determinations of h (e.g., ref. [46] Table 2, page 16) yield values $h = 0.68$, $\Omega_\Lambda = 0.69$. However, these are indirect, model-dependent and early universe determinations of h and Ω_Λ. The difference between the determinations of h in the late and in the early universe is an important problem on its own, e.g., ref. [82], although we do not treat this problem here.

Bosons give a positive contribution to dark energy through the cosmological quantum vacuum, while fermions give a negative contribution. Therefore, the boson contribution must dominate.

As discussed in Section 8, the lightest neutrino certainly contributes to the cosmological quantum vacuum unless it dissipates. Definitely, a boson contribution is needed. The photon and graviton contributions are irrelevant since their masses are most probably zero and at most $m_\gamma < 6 \times 10^{-17}$ eV, $\quad m_{\text{graviton}} < 4.7 \times 10^{-23}$ eV [83].

Massless particles contribute to the energy-momentum tensor through the trace anomaly [76,77]. This contribution is of the order of H_0^4, where H_0 is the Hubble parameter today:

$$H_0 = 1.558 \times 10^{-33}\ \text{eV}. \tag{91}$$

As a consequence, the massless particles' contribution to the energy-momentum tensor is exceedingly small to explain the observed value of the dark energy.

- A scalar particle can produce the dark energy today Equation (89) through its quantum cosmological vacuum provided:
- Its mass is of the order of 1 meV, and it is very weakly coupled.
- Its lifetime is of the order of the age of the universe.

Spontaneous symmetry breaking of continuous symmetries produces massless scalars as Goldstone bosons. If, in addition, this continuous symmetry is slightly violated, the Goldstone boson acquires a small mass. This is the natural mechanism that generates light scalars, and several particles have been proposed on these grounds in the past. The axion is certainly the one that has caught more attention in the literature. Other proposed particles are the familons and the majorons [56,84–89].

The (invisible) axion [51–55] (if it exists) is hence a candidate to be the source of dark energy.

Axions were proposed to solve the strong CP problem in QCD [47–50]. Axions acquire a mass after the breaking of the Peccei–Quinn (PQ) symmetry when the temperature of the universe was at the PQ symmetry breaking scale$\sim f_a$ [62–66]. All axion couplings are inversely proportional to f_a, and the axion mass is given by

$$M_a \;\simeq\; 6 \;\times\; \left(\frac{10^9 \text{ GeV}}{f_a}\right) \text{meV}. \tag{92}$$

The following range ('axion window') is currently acceptable for the axion mass [63,73–75,90–92]:

$$10^{-3} \;\; \text{meV} \;\lesssim\; M_a \;\lesssim\; 10 \;\; \text{meV} \,. \tag{93}$$

Therefore, this pseudoscalar particle has extremely weak coupling to gluons and quarks, and hence it contributes to the cosmological quantum vacuum. For example, the axion–photon–photon coupling is given by

$$g_{a\,\gamma\,\gamma} \sim \frac{10^{-10}}{\text{GeV}} \left(\frac{M_a}{1 \;\; \text{meV}}\right) . \tag{94}$$

As a consequence, the axion lifetime to decay into photons is much longer than the age of the universe. Dissipation of the energy in the cosmological quantum axion vacuum takes longer than the age of the universe too.

- An axion with mass$\sim$1 meV and hence $f_a \sim 10^9$ GeV decouples from the plasma at a scale of energies$\sim 2 \times 10^5$ GeV , that is, at redshift $z_S \sim 2.2 \times 10^{18}$. The temperatures of the axions and neutrinos today are lower than that of photons today,

$$T_{\nu \text{ today}} \;=\; \left(\frac{4}{11}\right)^{\frac{1}{3}} T_{\text{CMB today}} \;=\; 0.1676 \;\; \text{meV}, \quad T_{a\,\text{today}} \;=\; 0.078 \;\; \text{meV}. \tag{95}$$

  Because the axion lifetime is of the order or larger than the age of the universe, no specific properties of the axion play a role in dark energy, except for its mass and decoupling redshift. However, the dark energy depends on the decoupling redshift rather weakly because it is through its logarithm (see Equation (77)).
- Neutrinos in the universe are believed to be effectively free particles when the temperature of the universe is below$\sim$1 MeV. That is, neutrinos decouple at a redshift $z_F \sim 0.6 \times 10^{10}$. Before such time, electrons and neutrinos interacted, keeping them in thermal equilibrium.
- Therefore, we can treat the axion with mass$\sim$1 meV and the lightest neutrino as free particles in the universe for redshifts $z < z_S \sim 2.2 \times 10^{18}$ and $z < z_F \sim 0.6 \times 10^{10}$, respectively.

8. Neutrino Mass Eigenstates

As is known, the two heavier neutrino mass eigenstates ν_2 and ν_3 with masses m_2 and m_3, respectively, annihilate with their respective anti-neutrinos, yielding the lightest neutrino eigenstate ν_1 and its antiparticle through weak interactions. However, this process is too slow for nonrelativistic neutrinos even compared with the age of the universe. Their decay rates can be estimated to be

$$\Gamma_2 \sim G_F^2 \, m_2^5 \sim \frac{1}{1.5 \times 10^{33} \text{ yr}}, \quad \Gamma_3 \sim G_F^2 \, m_3^5 \sim \frac{1}{5 \times 10^{29} \text{ yr}},$$

where $G_F = 1.166 \times 10^{-23} \, (\text{eV})^{-2}$ stands for the Fermi coupling.

Neutrinos with masses $m_2 \sim 0.01$ eV or $m_3 \sim 0.05$ eV will produce through their cosmological quantum vacuum today a large negative contribution to dark energy.

Therefore, the heavier neutrinos (ν_2 and ν_3) must annihilate with their respective anti-neutrinos into the lightest neutrino ν_1 through a mechanism such that

$$\Gamma_3 \gtrsim \Gamma_2 \gtrsim (\text{age of the universe})^{-1}. \tag{96}$$

Our results for dark energy are independent of the details of the decay mechanism. All that counts is that the decay rates of the heavier neutrinos obey Equation (96).

As a minimal assumption, let us consider the following effective couplings between the neutrinos,

$$\frac{1}{M'^{\,2}} \bar{\Psi}_2 \, \Psi_2 \, \bar{\Psi}_1 \, \Psi_1 \quad , \quad \frac{1}{M'^{\,2}} \bar{\Psi}_3 \, \Psi_3 \, \bar{\Psi}_1 \, \Psi_1, \tag{97}$$

where M' is a mass scale much larger than the neutrino masses. We thus find,

$$\Gamma_2 \sim \frac{(m_2)^5}{M'^{\,4}} \quad , \quad \Gamma_3 \sim \frac{(m_3)^5}{M'^{\,4}}.$$

Imposing Equation (96) yields,

$$M' \lesssim 1 \text{ MeV} \;\; \text{for } m_2 = 0.01 \text{ eV} \quad \text{and} \quad M' \lesssim 10 \text{ MeV} \;\; \text{for } m_3 = 0.05 \text{ eV} . \tag{98}$$

The first estimated bound (1 MeV) applies for a direct hierarchy of neutrino masses ($m_3 \sim 0.05$ eV $> m_2 \sim 0.01$ eV $> m_1$), while the second estimate (10 MeV) is for an inverse hierarchy of neutrino masses ($m_3 \sim m_2 \sim 0.05$ eV $> m_1$).

Effective couplings of the type in Equation (97) can be obtained from different renormalizable models.

Notice that the two heavier neutrino decays contribute to the background of lighter neutrino particles but not to the neutrino quantum vacuum.

Lagrangians leading to effective couplings analogous to Equation (97) have been considered in the context of models to generate neutrino masses and to provide light dark matter candidates [93–104]. Moreover, mass ranges compatible with Equation (98) have been obtained from various and independent considerations [95–105]. This value also follows by setting $Q = 0$ (neutrinos has no charge) in Equation (1) of Ref. [69]. In case the effective couplings of Equation (97) arise from Yukawa couplings of the neutrinos with a scalar particle of mass M', this scalar particle cannot be a dark matter candidate since it decays into neutrino–antineutrino pairs.

The lightest neutrino with mass m_1 can be self-coupled through the interaction

$$\frac{1}{M''^{\,2}} \, (\bar{\Psi}_1 \, \Psi_1)^2.$$

Its decay rate,

$$\Gamma_1 \sim \frac{(m_1)^5}{M''^{\,4}},$$

is of the order or larger than the age of the universe when

$$M'' \lesssim \left(\frac{m_1}{\mathrm{meV}}\right)^{\frac{5}{4}} 50\ \mathrm{keV}. \tag{99}$$

Hence, if Equation (99) is fulfilled, the energy in the neutrino vacuum dissipates into the lightest neutrino's ν_1, thus contributing to the neutrino background.

9. Light Particle Masses and the Dark Energy Density Today

Let us consider the case where only one light scalar field and one light fermion field contribute to the quantum vacuum energy. That is, a light scalar and the lightest neutrino. We obtain from Equation (80) for the mass of the scalar,

$$M = \frac{2^{\frac{5}{4}}\sqrt{\pi}\,\rho_\Lambda^{\frac{1}{4}}}{\left(\log z_S - \frac{1}{4}\right)^{\frac{1}{4}}}\left[1 + \frac{\mathcal{N}\,m^4}{(4\,\pi)^2\,\rho_\Lambda}\left(\log z_F - \frac{1}{4}\right)\right]^{\frac{1}{4}}, \tag{100}$$

where we identified the vacuum energy density today $< \mathcal{H} > (0)$ with the observed dark energy ρ_Λ.

We now obtain using the observed value of the dark energy Equation (89) and the decoupling redshift for the neutrino $z_F \sim 0.6 \times 10^{10}$,

$$M = \frac{10.1\ \mathrm{meV}}{\left(\log z_S - \frac{1}{4}\right)^{\frac{1}{4}}}\left[1 + \mathcal{N}\left(\frac{m}{3.90\ \mathrm{meV}}\right)^4\right]^{\frac{1}{4}}. \tag{101}$$

If the lightest neutrino has a very small mass $m \ll 1$ meV or if it decays in the time scale of the age of the universe (see Equation (99)) so the neutrino vacuum dissipates, there is no neutrino contribution to the dark energy. In these cases, Equation (101) gives for the mass of the scalar:

$$M = \frac{10.1\ \mathrm{meV}}{\left(\log z_S - \frac{1}{4}\right)^{\frac{1}{4}}} : \quad \text{no vacuum neutrino energy.} \tag{102}$$

Assuming the scalar field to be the axion, we can use the value $z_S \sim 2.2 \times 10^{18}$ for the axion decoupling redshift, and Equation (101) becomes,

$$M(m) = 3.96\ \mathrm{meV}\left[1 + \mathcal{N}\left(\frac{m}{3.90\ \mathrm{meV}}\right)^4\right]^{\frac{1}{4}}. \tag{103}$$

The values of the neutrino masses are not yet known, only their differences are experimentally constrained. Both in the direct and inverse mass hierarchies, the mass m of the lightest neutrino can be in the meV range (or even zero).

According to ref. [69], we have

$$m = \frac{1}{3}\,m_2,$$

where m_2 is the mass of the middle neutrino. Combining this with the known neutrino mass differences yields

$$m = 3.2 \pm 0.1\ \mathrm{meV}. \tag{104}$$

This value for the neutrino mass perfectly agrees in order of magnitude with the see-saw prediction,

$$\frac{M^2{}_{\mathrm{Fermi}}}{M_{\mathrm{GUT}}} \simeq 6 \times 10^{-3}\ \mathrm{eV},$$

for the typical values $M_{\,\rm Fermi} \;=\; 250$ GeV and $M_{\,\rm GUT} \;=\; 10^{16}$ GeV of the Fermi and Grand Unified energy scales, respectively.

Equations (103) and (104) give for the axion mass:

$$M\;(m = 3.2\;\;\text{meV},\, \mathcal{N} = 1) \;=\; 4.35\;\;\text{meV} \quad , \quad M\;(m = 3.2\;\;\text{meV},\, \mathcal{N} = 2) \;=\; 4.66\;\;\text{meV}, \tag{105}$$

for Majorana and Dirac neutrinos, respectively.

If the lightest neutrino has a very small mass $m_1 \ll 1$ meV or if it decays in the time scale of the age of the universe (see Equation (99)), e.g., there is no neutrino contribution to the dark energy, then the axion mass is given by

$$M \;=\; 3.96\;\;\text{meV}: \quad \text{no vacuum neutrino energy.} \tag{106}$$

All the axion mass value Equation (103), (105) and (106) found here describe the dark energy observed today Equation (89). The numerical values for the axion mass in Equations (105) and (106) are within the astrophysical bound of Equation (93).

We compute the equation of state today from Equation (80) and display it in Table 1 in three relevant cases: (i) no neutrino contribution to the dark energy, (ii) a Majorana neutrino contribution and (iii) a Dirac neutrino contribution. In all three cases, the observed value Equation (89) of the dark energy is imposed. For the last two cases, we choose the neutrino mass $m \;=\; 3.2$ meV and the scalar mass M given by Equation (105), e.g., 4.35 meV and 4.66 meV, respectively.

We see that $w(0)$ is slightly below -1 by an amount ranging from (-1.5×10^{-3}) to (-8×10^{-3}).

It can be noticed that the mass of the lightest neutrino (Equation (104)) turns to be much higher than today's neutrino temperature:

$$\frac{m_{\,\rm Dirac}}{T_{\,\nu\;\rm today}} \;=\; 19.6, \quad \frac{m_{\,\rm Majorana}}{T_{\,\nu\;\rm today}} \;=\; 23.3, \tag{107}$$

where we used Equation (95). That is to say, the neutrinos forming the neutrino background are, today, non-relativistic particles.

Let us now analyze the possibility in which all three neutrino eigenstates contribute to dark energy. This contribution crucially depends on the values of their masses to the power four through the dimensionless factor

$$\mathcal{F} \;\equiv\; \frac{1}{3\,(4\,\pi)^2}\,\frac{\sum_{\,i} m_i^4}{\rho_\Lambda},$$

as we see from Equations (85)–(87).

For the normal hierarchy, we have

$$m_1 \;=\; 3.2\;\;\text{meV}, \quad m_2 \;=\; 9.5\;\;\text{meV}, \quad m_3 \;=\; 47\;\;\text{meV},$$

and for the inverted hierarchy:

$$m_1 \;=\; 3.2\;\;\text{meV}, \quad m_2 \;=\; 47\;\;\text{meV}, \quad m_3 \;=\; 48\;\;\text{meV} \quad .$$

Thus, using Equation (89), the factor $\mathcal{F}$ takes the values

$$\mathcal{F}_{\rm normal} \;=\; 315, \quad \mathcal{F}_{\rm inverted} \;=\; 656.$$

Inserting these numbers in the equation of state today Equation (87) yields values for $w(0)$ in strong disagreement with the data unless we fine tune $z_S \simeq z_F$. Because there is no reason to have such equality, we conclude that the vacuum of the two heavier neutrinos must not contribute to the dark energy. Their quantum vacuum must dissipate, as discussed in Section 8.

10. The Future Evolution of the Universe

The future evolution of the universe follows by inserting the total energy density in the Einstein–Friedmann equation

$$H^2(t) \; = \; \frac{8\,\pi\,G}{3}\,\mathcal{H}_T,$$

where we use cosmic time t, G is the gravitational constant and the total energy density $\mathcal{H}_T$ is the sum of the contributions from the dark energy, the matter and the radiation.

We obtain using the dark energy expression Equation (83) the self-consistent Einstein–Friedmann evolution equation,

$$H^2(t) \; = \; H_0^2 \left[\Omega_\Lambda \left(1 + \beta_{\mathcal{N}} \, \log \frac{a(t)}{a_0} \right) \; + \; \Omega_{\text{matter}} \, \frac{a_0^3}{a^3(t)} \; + \; \Omega_{\text{rad}} \, \frac{a_0^4}{a^4(t)} \right], \tag{108}$$

where $a_0 \equiv a(\text{today})$, $\beta_{\mathcal{N}}$ is defined by Equation (84), $\rho_\Lambda = \rho_{\text{crit}}\,\Omega_\Lambda$ is given by Equation (9) and H_0 is the Hubble parameter today, being

$$\rho_{\text{crit}} \; = \; \frac{(3\,H_0^2)}{(8\,\pi\,G)}, \qquad H_0 \; = \; \frac{h}{[\,9.77813 \text{ Gyr}\,]}, \qquad \Omega_\Lambda \; = \; 0.76 \; = \; (1 \; - \; \Omega_{\text{matter}} \; - \; \Omega_{\text{rad}}). \tag{109}$$

We use the explicit values for M and m for Equations (104)–(106):

$$\beta_0 \; = \; 0.0238 : \quad \text{No vacuum neutrino energy;} \qquad \beta_1 \; = \; 0.0347 : \quad \text{Majorana neutrino;}$$

$$\beta_2 \; = \; 0.0459 : \quad \text{Dirac neutrino.}$$

For $a(t) \; \gtrsim \; a_0$, the matter and radiation contributions can be neglected in Equation (108), and we have,

$$\left[\frac{d\,\log a(t)}{d\,t} \right]^2 \simeq \; H_0^2\,\Omega_\Lambda \left[1 + \beta_{\mathcal{N}} \, \log \frac{a(t)}{a_0} \right].$$

This equation can be immediately integrated with the solution

$$a(t) \quad \overset{H_0\,t \,\gtrsim\, 1}{\simeq} \quad a_0 \; \exp\,[\,c_1\,H_0\,t \; + \; c_2\,(H_0\,t)^2\,], \tag{110}$$

where

$$c_1 \; = \; \sqrt{\Omega_\Lambda} \; = 0.87, \quad c_2 \; = \; \frac{1}{4}\,\Omega_\Lambda\,\beta_{\mathcal{N}} \; = \; 0.19\;\beta_{\mathcal{N}},$$

$$0.00452 \; < \; c_2 \; < \; 0.00872. \tag{111}$$

The left and right ends of the interval in c_2 Equation (111) correspond to the cases in which there is no neutrino contribution and to the lightest neutrino being a Dirac fermion with mass $m = 3.2$ meV, respectively.

We find that the Universe is presently reaching an asymptotic phase where it expands as indicated by Equation (110).

Equation (110) shows that the expansion of the Universe in the future is faster than in the de Sitter Universe.

Notice that the time scale of the accelerated expansion is huge,$\sim(1/H_0) = 13.4$ Gyr. The quadratic term dominates over the linear term in the exponent of Equation (110) by a time $t{\sim}(100/H_0)$ to $(200/H_0)$.

In this accelerated universe, Equation (108) shows that the Hubble radius $(1/H)$ decreases with time as

$$\frac{1}{H} \sim \frac{1}{H_0\,\sqrt{\log a(t)}}.$$

11. Discussion

The non-trivial energy and pressure that we have is an effect resulting from the expansion of space-time as it arises from the $\log a(\eta)$ factor in Equation (73). No dark energy appears in Minkowski space-time. Namely, the formation and growth of the vacuum density, the vacuum energy density and pressure is an effect due to the presence of quantum fields in an expanding cosmological space-time.

Notice that the energy scale of the cosmological vacuum is given by the mass of the particle when this mass is larger than the Hubble constant (see Equation (91)). For massless particles, the energy scale of the cosmological vacuum is given by the Hubble constant.

The axion evolution for $z \geq 10^{18}$ as well as the neutrino evolution for $z \geq 10^{10}$ are beyond the scope of this article. Namely, the regime where the interaction of axions and neutrinos with the plasma particles cannot be neglected. We choose as the initial state for both the axions and the neutrinos the vacuum thermal equilibrium state. It must be remarked that the vacuum energy at late times is independent of the initial temperature, as shown by Equation (70).

Before decoupling, particle interaction is non-negligable and dissipation is important for depleting the vacuum energy [41,106]. Hence, the vacuum energy can only become significant after decoupling. Therefore, it is a good approximation to just study the free quantum field evolution in cosmological space-time after decoupling.

The initial conditions Equations (26) and (50) are imposed at the origin of the conformal time. We shall see now that they are equivalent to the Bunch–Davies vacuum conditions. Since the initial time corresponds to a large value of redshift, it corresponds to asymptotic times in the past in a very good approximation. More precisely, the conformal time is related to the redshift z by

$$\begin{aligned} \eta &= \frac{3\,t_0}{\sqrt{1\,+\,z}} : \quad \text{matter}-\text{dominated era}, \\ \eta &= \frac{2\,t_0\,\sqrt{1\,+\,z_{\text{eq}}}}{1\,+\,z} + \frac{t_0}{\sqrt{1\,+\,z_{\text{eq}}}} : \quad \text{radiation}-\text{dominated era}\,, \end{aligned} \tag{112}$$

where $t_0 = 13.7$ Gyr is the age of the universe and $1+z_{\text{eq}} = 3048$ is the transition from the radiation-dominated to the matter-dominated era. $\eta_0 = 3\,t_0$ corresponds to the present time. For $z \gg z_{\text{eq}}$, we see that,

$$\eta \simeq \frac{t_0}{\sqrt{1+z_{\text{eq}}}} = 0.018\,t_0.$$

Hence, the conformal time at decoupling differs from the conformal time today $\eta_0 = 3\,t_0$ by an amount$\sim 3\,t_0$. As a result, the initial time can be considered as an asymptotic time deep in the past. More precisely, the change in the phases of the mode functions is characterized by $(M\,t_0) \sim 3\times 10^{30}$ for a typical mass $M\sim 4$ meV. Hence, the initial conditions for the mode functions Equations (26) and (50) are virtually identical to Bunch–Davies initial conditions.

The vacuum density and energy density Equations (69) and (73) are determined by the short distance behavior of the two-point function in coordinate space. In momentum space, it is the high energy behavior that dominates the vacuum density and energy density for late times. The physical quantities can be written as integrals of mode functions, as in Equations (24) and (40). One can see that the relevant comoving momenta k values contributing on a physical energy scale q take the value $k = q\;a(\eta)$. At late times (e.g., today), $a(\eta) \sim z_{\text{ decoupling}}$, therefore only large $k \sim z_{\text{decoupling}}\;M$ are relevant. This fact decreases the effect of the initial conditions. Analogous effects take place for the initial conditions of inflationary fluctuations with the exception of the low multipoles, particularly the quadrupole [107–111].

- In Figure 1, we plot the equation of state $w(z)$ as a function of z for the three cases explicitly calculated in this paper:

- (i) No neutrino contribution to the dark energy and the scalar mass $M = 3.96$ meV.
- (ii) A Majorana neutrino with mass $m = 3.2$ meV and the scalar mass $M = 4.35$ meV.
- (iii) A Dirac neutrino with mass $m = 3.2$ meV and the scalar mass $M = 4.66$ meV (see the discussion in Section 9).
- We see that the equation of state in all the three cases (i)–(iii) differs from the cosmological constant case $w = -1$ by less than 1%.
- The value of the lightest neutrino mass Equation (104) is well below the neutrino mass splittings $\sqrt{\Delta m^2_{sun}}$ and $\sqrt{\Delta m^2_{atm}}$ and consistent with both direct and inverse mass hierarchies. A quasi-degenerate mass spectrum will give a large negative contribution to the dark energy and will require a scalar particle with a mass $M \gtrsim 100$ meV to reproduce the observed dark energy data Equation (89). Such a particle can very well exist, but it cannot be the axion (see Equation (93)). Indeed, the scalar particle can have the mass value given by Equation (106) in case all three neutrinos decay in a time scale of the age of the universe in order to dissipate their cosmological vacuum energy, as discussed in Section 8.
- On the other hand, a range of neutrino masses from 10^{-3} eV to 0.1 eV in agreement with neutrino mass differences from oscillations and the value Equation (104) for the mass of the lightest neutrinos is compatible with a consistent baryogenesis.

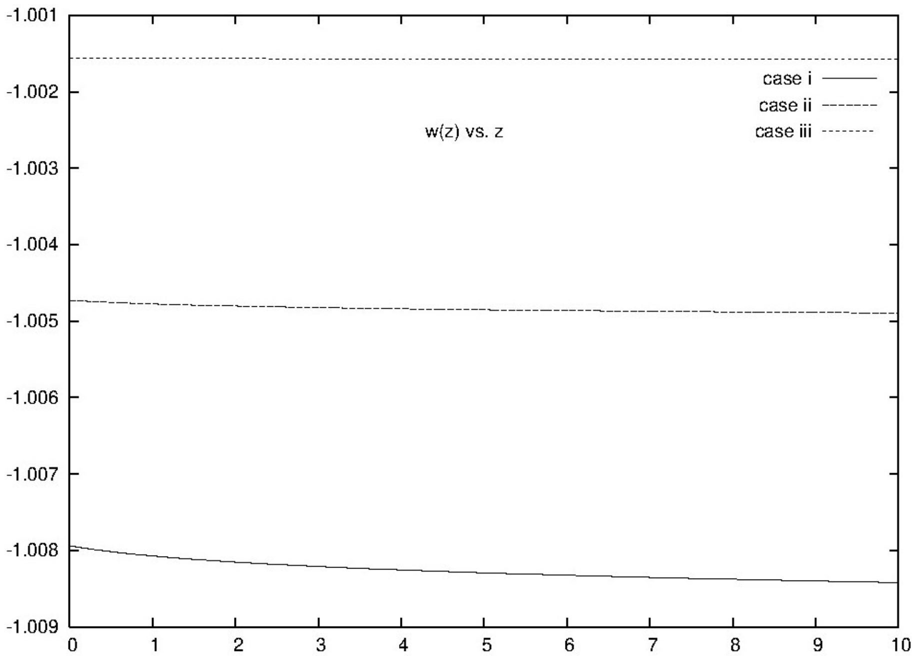

Figure 1. The equation of state *w(z)* vs. the redshift z for the three cases explicitly calculated in this paper: (i) [full line] No neutrino contribution to the dark energy and the scalar mass $M = 3.96$ meV. (ii) [broken line] A Majorana neutrino with mass $m = 3.2$ meV and the scalar mass $M = 4.35$ meV. (iii) [dotted line] A Dirac neutrino with mass $m = 3.2$ meV and the scalar mass $M = 4.66$ meV. (See the discussion in Section 9.) In all three cases, $w < -1$ by less than 1%.

12. Conclusions

- We find that the presence of a cosmological quantum vacuum energy with an equation of state just below -1 is the unavoidable consequence of the existence of light particles with very weak couplings. Bosons yield positive contributions and fermions yield negative contributions to the vacuum energy.
- It must be noticed that there is a present lack of knowledge about the low-energy (energy$\sim$1 meV) particle physics region. Actually, most of the constraints on this sector follow from astrophysics and cosmology, including the new constraints that we obtain here on the axion mass.

- No exotic physics need to be invoked to explain the dark energy. Since the observed energy scale of the dark energy is very low, we find it natural to explain it only through low-energy physics. The effects from energy scales higher than 1 eV or even 1 MeV arrive strongly suppressed to the dark energy scale of 1 meV.
- In summary, dark energy can be explained by a very light and very weakly coupled scalar particle, which decouples by redshift $z_S \gg 1$. If the scalar particle is the axion, then $z_S \sim 2.2 \times 10^{18}$.
  We have four main cases:
  (i) No neutrino contribution. This happens when the lightest neutrino has a mass $m \ll 1$ meV and when the vacuum neutrino contribution dissipates in the time scale of the age of the universe (see Equation (99)). The scalar mass must be

$$M = \frac{10.1 \ \text{meV}}{\left(\log z_S - \frac{1}{4} \right)^{\frac{1}{4}}} : \quad \text{no vacuum neutrino energy.} \tag{113}$$

  If the scalar is the axion, then $M = 3.96$ meV in this case.
  (ii) The lightest neutrino is Majorana and has a mass $m \simeq 3.2$ meV. Then, the scalar mass must be

$$M = \frac{11.1 \ \text{meV}}{\left(\log z_S - \frac{1}{4} \right)^{\frac{1}{4}}} : \quad \text{the Majorana neutrino contributes.}$$

  If the scalar is the axion, then $M = 4.35$ meV in this case.
  (iii) The lightest neutrino is Dirac and has a mass $m \simeq 3.2$ meV. Then, the scalar mass must be

$$M = \frac{11.9 \ \text{meV}}{\left(\log z_S - \frac{1}{4} \right)^{\frac{1}{4}}} : \quad \text{the Dirac neutrino contributes.}$$

  If the scalar is the axion, then $M = 4.66$ meV in this case.
- Therefore, in all the three cases (i)–(iii) above where the axion explains the dark energy, we predict its mass in the range:

$$3.96 \ \text{meV} < M < 4.66 \ \text{meV}. \tag{114}$$

  The left and right ends of the interval in Equation (114) correspond to no neutrino contribution and to the lightest neutrino as a Dirac fermion with mass $m = 3.2$ meV, respectively.
- In short, we uncovered here the general mechanism producing the dark energy today. This mechanism has it grounds in well-known quantum physics and cosmology. The dark energy appears as a quantum vacuum effect arising when stable and weakly coupled quantum fields live in expanding cosmological space-times. That is to say, dark energy in the universe today is a QFT effect in (classical) curved space-times. That is to say, this is a semiclassical gravity effect.
- In addition, we have found here that the axion with mass in the meV range is a very serious candidate for dark energy, while we have shown already [112,113] that it is robustely excluded as a dark matter candidate. The cosmic dark energy today is on the meV scale, while the dark matter (cosmic and galactic) particle is on the keV scale [113–120].
- Many research avenues are open now connecting dark energy and light particles physics. The more immediate being:

(1) The study of the radiative corrections to the axion and neutrino cosmological vacuum evolution from their interactions.
(2) The study of the early neutrino and axion dynamics at temperatures $\gtrsim 1$ MeV and $\gtrsim 10^6$ GeV, respectively.
(3) The study of particle propagation in the media formed by the axion and the neutrino vacuum.
(4) Last but not least: The probable deep connection between dark energy and dark matter through low-energy particle states beyond the standard model of particle physics.

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Appendix A. Dimensional and Cutoff Regularization of the Vacuum Energy

Physical vacuum quantities are computed in Section 4 as the equal point limit of two-point functions. The distance z between the points Equation (61) naturally plays the role of the regularization parameter. Alternatively, one can regularize the two-point function with dimensional regularization or cutoff regularization and set $z = 0$ in the regularized expressions.

In dimensional regularization, we have

$$G_\epsilon(M\,a, a) \;\equiv\; < T\;\varphi(\vec{x}, \eta)\;\varphi(\vec{x}, \eta) > \;=\; \frac{1}{a^2} \int \frac{d^{4-2\,\epsilon}k}{(2\,\pi)^{4-2\,\epsilon}} \;\frac{i}{k^2 - a^2\,M^2 + i\,0} \tag{A1}$$

$$= \frac{M^{2-2\,\epsilon}}{(4\,\pi)^{2-\epsilon}}\;\frac{1}{a^{2\,\epsilon}}\;\Gamma(\epsilon - 1). \tag{A2}$$

Subtracting the value in Minkowski space-time ($a = 1$) yields,

$$G_\epsilon(M\,a, a) - G_\epsilon(M, 1) \;=\; M^{(2-2\,\epsilon)}\;\frac{\Gamma(\epsilon-1)}{(4\,\pi)^{(2-\epsilon)}}\left[a^{-2\,\epsilon} - 1\right] \overset{\epsilon\to 0}{=} \frac{M^2}{2\,(2\,\pi)^2}\;\log a, \tag{A3}$$

in agreement with Equation (68).

Alternatively, by regularizing with an ultraviolet cutoff Λ in four space-time dimensions, we have

$$G_\Lambda(M\,a) \;\equiv\; \frac{1}{a^2} \int \frac{d^4k}{(2\,\pi)^4}\;\frac{i}{k^2 - a^2\,M^2 + i\,0} \;=\; \left(\frac{\Lambda}{4\,\pi\,a}\right)^2 - \left(\frac{M}{4\,\pi}\right)^2 \log\left[1 + \left(\frac{\Lambda}{M\,a}\right)^2\right] =$$
$$\overset{\Lambda\to\infty}{=} \left(\frac{\Lambda}{4\,\pi\,a}\right)^2 - \frac{M^2}{2\,(2\,\pi)^2}\;\log\left[\frac{\Lambda}{M\,a}\right]. \tag{A4}$$

Subtracting the divergence in $\Lambda = \infty$ again leads to the result Equations (A3) and (68):

$$G_\Lambda(M\,a) - \left[\left(\frac{\Lambda}{4\,\pi\,a}\right)^2 - \frac{M^2}{2\,(2\,\pi)^2}\;\log\frac{\Lambda}{M}\right] \overset{\Lambda\to\infty}{=} \frac{M^2}{2\,(2\,\pi)^2}\;\log a.$$

We have therefore verified that the point splitting regularization used in Section 4 as well as dimensional and cutoff regularization methods yield identical results. (It has been known for a long time that dimensional regularization gives the same physical results as other regularization methods [121–124]). Analogous results are valid for the two-point fermion function.

References

1. Riess, A.G.; Filippenko, A.V.; Challis, P.; Clocchiatti, A.; Diercks, A.; Garnavich, P.M.; Gilliland, R.L.; Hogan, C.J.; Jha, S.; Kirshner, R.P.; et al. Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. *Astron. J.* **1998**, *116*, 1009. [CrossRef]
2. Garnavich, P.; Jha, S.; Challis, P.; Clocchiatti, A.; Diercks, A.; Filippenko, A.V.; Gilliland, R.L.; Hogan, C.J.; Kirshner, R.P.; Leibundgut, L.B.; et al. Supernova Limits on the Cosmic Equation of State. *Astrophys. J.* **1998**, *509*, 74. [CrossRef]
3. Perlmutter, S.; Aldering, G.; Goldhaber, G.; Knop, R.A.; Nugent, P.; Castro, P.G.; Deustua, S.; Fabbro, S.; Goobar, A.; Groom, D.E.; et al. Measurements of Ω and Λ from 42 High-Redshift Supernovae. *Astrophys. J.* **1999**, *517*, 565. [CrossRef]
4. Schmidt, B.P. Measuring global curvature and Cosmic acceleration with Supernovae. In *Phase Transitions in Cosmology: Theory and Observations*; de Vega, H.J., Khalatnikov, I.M., Sanchez, N.G., Eds.; NATO ASI Series; Kluwer Publishing: Philadelphia, PA, USA, 2001; Volume 40, pp. 249–266.
5. Knop, R.A. et al. [The Supernova Cosmology Project] New Constraints on ΩM, $\Omega\Lambda$, and w from an Independent Set of 11 High-Redshift Supernovae Observed with the Hubble Space Telescope. *Astrophys. J.* **2003**, *598*, 102. [CrossRef]
6. Riess, A.G.; Strolger, L.-G.; Tonry, J.; Casertano, S.; Ferguson, H.C.; Mobasher, B.; Challis, P.; Filippenko, A.V.; Jha, S.; Li, W.; et al. Type Ia Supernova Discoveries at z >1 from the Hubble Space Telescope: Evidence for Past Deceleration and Constraints on Dark Energy Evolution. *Astrophys. J.* **2004**, *607*, 665. [CrossRef]
7. Astier, P.; Guy, J.; Regnault, N.; Pain, R.; Aubourg, E.; Balam, D.; Basa, S.; Carlberg, R.G.; Fabbro, S.; Fouchez, D.; et al. The Supernova Legacy Survey: measurement of ΩM, $\Omega\Lambda$, and w from the first year data set. *Astron. Astrophys.* **2006**, *447*, 31. [CrossRef]
8. Eisenstein, D.J.; Zehavi, I.; Hogg, D.W.; Scoccimarro, R.; Blanton, M.R.; Nichol, R.C.; Scranton, R.; Seo, H.J.; Tegmark, M.; Zheng, Z.; et al. Detection of the Baryon Acoustic Peak in the Large-Scale Correlation Function of SDSS Luminous Red Galaxies. *Astrophys. J.* **2005**, *633*, 560. [CrossRef]
9. Conley, A. et al. [The Supernova Cosmology Project] Measurement of ΩM, $\Omega\Lambda$ from a Blind Analysis of Type Ia Supernovae with CMAGIC: Using Color Information to Verify the Acceleration of the Universe. *Astrophys. J.* **2006**, *644*, 1. [CrossRef]
10. Wood-Vasey, W.M.; Miknaitis, G.; Stubbs, C.W.; Jha, S.; Riess, A.G.; Garnavich, P.M.; Kirshner, R.P.; Aguilera, C.; Becker, A.C.; Blackman, J.W.; et al. Observational Constraints on the Nature of Dark Energy: First Cosmological Results from the ESSENCE Supernova Survey. *Astrophys. J.* **2007**, *666*, 694–715. [CrossRef]
11. Miknaitis, G.; Pignata, G.; Rest, A.; Wood-Vasey, W.M.; Blondin, S.; Challis, P.; Smith, R.C.; Stubbs, C.W.; Suntzeff, N.B.; Foley, R.J.; et al. The ESSENCE Supernova Survey: Survey Optimization, Observations, and Supernova Photometry. *Astrophys. J.* **2007**, *666*, 674–693. [CrossRef]
12. Riess, A.G.; Strolger, L.G.; Casertano, S.; Ferguson, H.C.; Mobasher, B.; Gold, B.; Challis, P.J.; Filippenko, A.V.; Jha, S.; Li, W.; et al. New Hubble Space Telescope Discoveries of Type Ia Supernovae at z $\geq$ 1: Narrowing Constraints on the Early Behavior of Dark Energy. *Astrophys. J.* **2007**, *659*, 98–121. [CrossRef]
13. Abbott, T. et al. [Dark Energy Survey Collaboration] The Dark Energy Survey: More than dark energy—An overview. *MNRAS* **2016**, *460*, 1270.
14. Abbott, T.M.C. et al. [DES Collaboration] First Cosmology Results using Type Ia Supernovae from the Dark Energy Survey: Constraints on Cosmological Parameters. *Astrophys. J. Lett.* **2019**, *872*, L30. [CrossRef]
15. Abbott, T.M.C. et al. [DES Collaboration] Dark Energy Survey Year 3 Results: Cosmological Constraints from Galaxy Clustering and Weak Lensing. *Phys. Rev. D* **2022**, *105*, 023520. [CrossRef]
16. Euclid. Available online: https://sci.esa.int/web/euclid/ (accessed on 22 March 2023).
17. Legacy Survey of Space and Time LSST-Vera C. Rubin Observatory. Available online: https://www.lsst.org/ (accessed on 22 March 2023).
18. Peebles, P.J.E.; Ratra, B. The cosmological constant and dark energy. *Revs. Mods. Phys.* **2003**, *75*, 559. [CrossRef]
19. Padmanabhan, T. Cosmological constant—The weight of the vacuum. *Phys. Rept.* **2003**, *380*, 235. [CrossRef]
20. Copeland, E.J.; Sami, M.; Tsujikawa, S. Dynamics of Dark Energy. *Int. J. Mod. Phys.* **2006**, *D15*, 1753–1936. [CrossRef]
21. Nobbenhuis, S. Categorizing Different Approaches to the Cosmological Constant Problem. *Found. Phys.* **2006**, *36*, 613. [CrossRef]
22. Frieman, J.A.; Turner, M.S.; Huterer, D. Dark Energy and the Accelerating Universe. *Ann. Rev. Astron. Astrophys.* **2008**, *46*, 385–432. [CrossRef]
23. Huterer, D.; Shafer, D.L. Dark Energy Two Decades after: Observables, Probes, Consistency Tests. *Rep. Prog. Phys.* **2018**, *81*, 016901. [CrossRef]
24. Fardon, R.; Nelson, A.E.; Weiner, N. Dark energy from mass varying neutrinos. *J. Cosmol. Astropart. Phys.* **2004**, *410*, 5. [CrossRef]

25. Fardon, R.; Nelson, A.E.; Weiner, N.J. Supersymmetric theories of neutrino dark energy. *J. High Energy Phys.* **2006**, *3*, 042. [CrossRef]
26. Peccei, R.D. Neutrino models of dark energy. *Phys. Rev.* **2005**, *D71*, 023527. [CrossRef]
27. Hung, P.Q.R. Horvat, Mass-varying neutrinos from a variable cosmological constant. *J. Cosmol. Astropart. Phys.* **2006**, *1*, 15.
28. Das, S.; Weiner, N. Late forming dark matter in theories of neutrino dark energy. *Phys. Rev.* **2011**, *D84*, 123511. [CrossRef]
29. Guendelman, E.I.; Kaganovich, A.B. Neutrino generated dynamical dark energy with no dark energy field. *Phys. Rev.* **2013**, *D87*, 044021. [CrossRef]
30. Valentino, E.D.; Gariazzo, S.; Mena, O.; Vagnozzi, S. Soundness of dark energy properties. *J. Cosmol. Astropart. Phys.* **2020**, *07*, 045. [CrossRef]
31. Mortonson, M.J.; Weinberg, D.H.; White, M. Dark Energy: A Short Review. *arXiv* **2013**, arXiv:1401.0046.
32. Frusciante, N.; Perenon, L. Effective field theory of dark energy: A review. *Phys. Rept.* **2020**, *857*, 1–63. [CrossRef]
33. Sola Peracaula, J. The cosmological constant problem and running vacuum in the expanding universe. *Phil. Trans. R. Soc. A* **2022**, *380*, 20210182. [CrossRef]
34. Albrecht, A.; Bernstein, G.; Cahn, R.; Freedman, W.L.; Hewitt, J.; Hu, W.; Huth, J.; Kamionkowski, M.; Kolb, E.W.; Knox, L.; et al. Report of the Dark Energy Task Force, DoE, NASA and NSF. *arXiv* **2006**, arXiv:astro-ph/0609591.
35. Abdalla, E.; Abellan, G.F.; Aboubrahim, A.; Agnello, A.; Akarsu, O.; Akrami, Y.; Alestas, G.; Aloni, D.; Amendola, L.; Anchordoqui, L.A.; et al. Cosmology Intertwined: A Review of the Particle Physics, Astrophysics, and Cosmology Associated with the Cosmological Tensions and Anomalies. *J. High Energy Astrophys.* **2022**, *34*, 49–211
36. Sanchez, N.G. Quantum discrete levels of the Universe from the early trans-Planckian vacuum to the late dark energy. *Phys. Rev.* **2021**, *D104*, 123517. [CrossRef]
37. Sanchez, N.G. New quantum phase of the Universe before inflation and its cosmological and dark energy implications. *Int. J. Mod. Phys.* **2019**, *A34*, 1950155. [CrossRef]
38. Sanchez, N.G. The classical-quantum duality of nature including gravity. *Int. J. Mod. Phys.* **2019**, *D28*, 1950055. [CrossRef]
39. Boyanovsky, D.; de Vega, H.J.; Holman, R.; Lee, D.S.; Singh, A. Dissipation via particle production in scalar field theories. *Phys. Rev.* **1995**, *D51*, 4419. [CrossRef] [PubMed]
40. Boyanovsky, D.; de Vega, H.J.; Holman, R.; Salgado, J.F.J. Analytic and Numerical Study of Preheating Dynamics. *Phys. Rev.* **1996**, *D54*, 7570. [CrossRef] [PubMed]
41. Boyanovsky, D.; D'Attanasio, M.; de Vega, H.J.; Holman, R.; Lee, D.-S. Linear versus nonlinear relaxation: Consequences for reheating and thermalization. *Phys. Rev.* **1995**, *D52*, 6805. [CrossRef]
42. Baacke, J.; Pätzold, C. Renormalization of the nonequilibrium dynamics of fermions in a flat FRW universe. *Phys. Rev.* **2000**, *D62*, 084008. [CrossRef]
43. Boyanovsky, D.; Destri, C.; de Vega, H.J.; Holman, R.; Salgado, J.F.J. Asymptotic dynamics in scalar field theory: Anomalous relaxation. *Phys. Rev.* **1998**, *D57*, 7388. [CrossRef]
44. Yao, W.M. et al. [Particle Data Group]. Available online: http://pdg.lbl.gov/ (accessed on 21 March 2023).
45. Spergel, D.N.; Bean, R.; Doré, O.; Nolta, M.R.; Bennett, C.L.; Dunkley, J.; Hinshaw, G.; Jarosik, N.; Komatsu, E.; Page, L.; et al. Three-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Implications for Cosmology. *Astrophys. J.* **2007**, *170*, 377. [CrossRef]
46. Aghanim, N. et al. [Planck Collaboration] Planck 2018 results. VI. Cosmological parameters. *Astron. Astrophys.* **2020**, *641*, 67.
47. Peccei, R.D.; Quinn, H.R. CP Conservation in the Presence of Pseudoparticles. *Phys. Rev. Lett.* **1977**, *38*, 1440. [CrossRef]
48. Weinberg, S. A New Light Boson? *Phys. Rev. Lett.* **1978**, *40*, 223. [CrossRef]
49. Wilczek, F. Problem of Strong P and T Invariance in the Presence of Instantons. *Phys. Rev. Lett.* **1978**, *40*, 279. [CrossRef]
50. Co, R.T.; Hall, L.J.; Harigaya, K. Axion Kinetic Misalignment Mechanism. *Phys. Rev. Lett.* **2020**, *124*, 251802. [CrossRef] [PubMed]
51. Kim, J.E. Weak-Interaction Singlet and Strong CP Invariance. *Phys. Rev. Lett.* **1979**, 43, 103. [CrossRef]
52. Shifman, M.A.; Vainshtein, A.I.; Zakharov, V.I. Can confinement ensure natural CP invariance of strong interactions? *Nucl. Phys.* **1980**, *B166*, 493. [CrossRef]
53. Dine, M.; Fischler, W.; Srednicki, M. A simple solution to the strong CP problem with a harmless axion. *Phys. Lett.* **1981**, *B104*, 199. [CrossRef]
54. Zhitnitsky, A.R.; Sov, J. On Possible Suppression of the Axion Hadron Interactions. *Nucl. Phys.* **1980**, *31*, 260. (In Russian)
55. Kim, J.E.; Carosi, G. Axions and the strong CP problem. *Rev. Mod. Phys.* **2010**, *82*, 557–602. [CrossRef]
56. Chikashige, Y.; Mohapatra, R.N.; Peccei, R.D. Are there real Goldstone bosons associated with broken lepton number? *Phys. Lett. B* **1981**, *98*, 265. [CrossRef]
57. Dodelson, S. *Modern Cosmology*; Academic Press: San Diego, CA, USA, 2003.
58. Kolb, E.W.; Turner, M.S. *The Early Universe*; Addison Wesley: Redwood City, CA, USA, 1990.
59. Gorbunov, D.S.; Rubakov, V.A. Theory of the Early Universe I. In *Hot Big Bang Theory*; World Scientific: Singapore, 2011.
60. Valentino, E.D.; Melchiorri, A.; Mena, O. Dark radiation sterile neutrino candidates after Planck data. *J. Cosmol. Astropart. Phys.* **2013**, *11*, 18. [CrossRef]
61. Archidiacono, M.; Gariazzo, S. wo Sides of the Same Coin: Sterile Neutrinos and Dark Radiation, Status and Perspectives. *Universe* **2022**, *8*, 175. [CrossRef]
62. Peccei, R.D. The Strong CP Problem and Axions. *Lect. Notes Phys.* **2008**, *741*, 3–17.

63. Sikivie, P. Axion Cosmology. *Lect. Notes Phys.* **2008**, *741*, 19–50.
64. Marsh, D.J.E. Axion cosmology. *Phys. Rep.* **2016**, *643*, 1–79. [CrossRef]
65. Borsanyi, S.S.; Dierigl, M.; Fodor, Z.; Katz, S.D.; Mages, S.W.; Nogradi, D.; Redondo, J.; Ringwald, A.; Szabo, K.K. Axion cosmology, lattice QCD and the dilute instanton gas. *Phys. Lett. B* **2016**, *752*, 175. [CrossRef]
66. Ballesteros, G.; Redondo, L.; Ringwald, A.; Tamarit, C. Unifying Inflation with the Axion, Dark Matter, Baryogenesis, and the Seesaw Mechanism. *Phys. Rev. Lett.* **2017**, *118*, 071802. [CrossRef]
67. Balazs, C.; Bloor, S.; Gonzalo, T.E.; Handley, W.; Hoof, S.; Kahlhoefer, F.; Lecroq, M.; Marsh, D.J.E.; Renk, J.J.; Scott, P.; et al. Cosmological constraints on decaying axion-like particles: a global analysis. *J. Cosmol. Astropart. Phys.* **2022**, *12*, 27. [CrossRef]
68. Hlozek, R.; Grin, D.; Marsh, D.J.E.; Ferreira, P.G. A search for ultralight axions using precision cosmological data. *Phys. Rev. D* **2015**, *91*, 103512. [CrossRef]
69. Sirlin, A. A Comment on the mass ratios of elementary fermions. *Comm. Nucl. Part. Phys.* **1994**, *21*, 227.
70. Albright, C.H. Normal vs. inverted hierarchy in type I seesaw models. *Phys. Lett. B* **2004**, *599*, 285. [CrossRef]
71. Casas, J.A.; Ibarra, A.; Jimenez-Alburquerque, F. Hints on the high-energy seesaw mechanism from the low-energy neutrino spectrum. *J. High Energy Phys.* **2007**, *704*, 64. [CrossRef]
72. Guo, W.-L.; Xing, Z.-Z.; Zhou, S. Neutrino Masses, Lepton Flavor Mixing and Leptogenesis in the Minimal Seesaw Model. *Int. J. Mod. Phys.* **2007**, *E16*, 1–50. [CrossRef]
73. Raffelt, G.G. Astrophysical Axion Bounds. *Lect. Notes Phys.* **2008**, *741*, 51–71.
74. Keil, W.; Janka, H.-T.; Schramm, D.N.; Sigl, G.; Turner, M.S.; Ellis, J. Fresh look at axions and SN 1987A. *Phys. Rev.* **1997**, *D56*, 2419. [CrossRef]
75. Janka, H.-T.; Keil, W.; Raffelt, G.; Seckel, D. Nucleon Spin Fluctuations and the Supernova Emission of Neutrinos and Axions. *Phys. Rev. Lett.* **1996**, *76*, 2621. [CrossRef] [PubMed]
76. Birrell, N.D.; Davies, P.C.W. Quantum fields in curved space. In *Cambridge Monographs in Mathematical Physics*; Cambridge University Press: Cambridge, UK, 1982.
77. Boyanovsky, D.; de Vega, H.J.; Sanchez, N.G. Quantum corrections to the inflaton potential and the power spectra from superhorizon modes and trace anomalies. *Phys. Rev.* **2005**, *D72*, 103006 [CrossRef]
78. Itzykson, C.; Zuber, J.B. *Quantum Field Theory*; McGraw-Hill: New York, NY, USA, 1980.
79. Riess, A.G.; Casertano, S.; Yuan, W.; Macri, L.M.; Scolnic, D. Large Magellanic Cloud Cepheid Standards Provide a 1% Foundation for the Determination of the Hubble Constant and Stronger Evidence for Physics beyond ΛCDM. *Astrophys. J.* **2019**, *876*, 85. [CrossRef]
80. Riess, A.G.; Casertano, S.; Yuan, W.; Bowers, J.B.; Macri, L.; Zinn, J.C.; Scolnic, D. Cosmic distances calibrated to 1% precision with gaia EDR3 parallaxes and Hubble space telescope photometry of 75 Milky Way cepheids confirm tension with ΛCDM. *Astrophys. J. Lett.* **2021**, *908*, L6. [CrossRef]
81. Riess, A.G.; Yuan, W.; Macri, L.M.; Scolnic, D.; Brout, D.; Casertano, S.; Jones, D.O.; Murakami, Y.; Anand, G.S.; Breuval, L.; et al. A comprehensive measurement of the local value of the Hubble constant with 1 km s- 1 mpc- 1 uncertainty from the Hubble space telescope and the sh0es team. *Astrophys. J. Lett.* **2022**, *934*, L7. [CrossRef]
82. The H_0 Award, Surprises in the Expansion History of the Universe. Available online: https://chalonge-devega.fr/Riess-07Dec22.mp4 (accessed on 8 December 2022).
83. Abbott, B.P.; The LIGO Scientific Collaboration; Virgo Collaboration. Tests of general relativity with the binary black hole signals from the LIGO-Virgo catalog GWTC-1. *Phys. Rev. D* **2019**, *100*, 104036. [CrossRef]
84. Valle, J.W.F. Gauge Theories and the Physics of Neutrino Mass. *Prog. Part. Nucl. Phys.* **1991**, *26*, 91. [CrossRef]
85. Valle, J.W.F. Neutrinos in astroparticle physics. *AIP Conf. Proc.* **2006**, *878*, 369–384
86. Valle, J.W.F. Concluding talk at NOW 2006. *Nucl. Phys. Proc. Suppl.* **2007**, *168*, 413–422. [CrossRef]
87. Lesgourgues, J.; Pastor, S. Massive neutrinos and cosmology. *Phys. Rept.* **2006**, *429*, 307. [CrossRef]
88. Mohapatra, R.N.; Smirnov, A.Y. Neutrino Mass and New Physics. *Annu. Rev. Nucl. Part. Sci.* **2006**, *56*, 569. [CrossRef]
89. Hannestad, S. Primordial Neutrinos. *Annu. Rev. Nucl. Part. Sci.* **2006**, *56*, 137. [CrossRef]
90. Sikivie, P.; Yang, Q. Bose-Einstein Condensation of Dark Matter Axions. *Phys. Rev. Lett.* **2009**, *103*, 111301. [CrossRef]
91. Erken, O.; Sikivie, P.; Tam, H.; Yang, Q. Cosmic axion thermalization. *Phys. Rev.* **2012**, *D85*, 063520.
92. Crisosto, N.; Sikivie, P.; Sullivan, N.S.; Tanner, D.B.; Yang, J.; Rybka, G. ADMX SLIC: Results from a Superconducting LC Circuit Investigating Cold Axions. *Phys. Rev. Lett.* **2020**, *124*, 241101. [CrossRef]
93. Turner, M.S. Early-Universe Thermal Production of Not-So-Invisible Axions. *Phys. Rev. Lett.* **1987**, *59*, 2489 [CrossRef]
94. Massó, E.; Rota, F.; Zsembinszki, G. Axion thermalization in the early universe. *Phys. Rev.* **2002**, *D66*, 023004. [CrossRef]
95. Chacko, Z.; Hall, L.J.; Okui, T.; Oliver, S.J. CMB signals of neutrino mass generation. *Phys. Rev.* **2004**, *D70*, 085008. [CrossRef]
96. Chacko, Z.; Hall, L.J.; Oliver, S.J.; Perelstein, M. Late Time Neutrino Masses, the LSND Experiment, and the Cosmic Microwave Background. *Phys. Rev. Lett.* **2005**, *94*, 111801. [CrossRef] [PubMed]
97. Okui, T. Searching for composite neutrinos in the cosmic microwave background. *J. High Energy Phys.* **2005**, 09, 017. [CrossRef]
98. Hall, L.J.; Oliver, S.J. Why Are Neutrinos Light?—An Alternative. *Nucl. Phys. Proc. Suppl.* **2004**, *137*, 269. [CrossRef]
99. Davoudiasl, H.; Kitano, R.; Kribs, G.D.; Murayama, H. Models of neutrino mass with a low cutoff scale. *Phys. Rev.* **2005**, *D71*, 113004. [CrossRef]

100. Boehm, C.; Farzan, Y.; Hambye, T.; Palomares-Ruiz, S.; Pascoli, S. Is it possible to explain neutrino masses with scalar dark matter? *Phys. Rev.* **2008**, *D77*, 043516. [CrossRef]
101. Bell, N.F.; Pierpaoli, E.; Sigurdson, K. Cosmological signatures of interacting neutrinos. *Phys. Rev.* **2006**, *D73*, 063523. [CrossRef]
102. Farzan, Y. Bounds on the coupling of the Majoron to light neutrinos from supernova cooling. *Phys. Rev.* **2003**, *D67*, 073015. [CrossRef]
103. Baker, J.L.; Goldberg, H.; Perez, G.; Sarcevic, I. Probing late neutrino mass properties with supernova neutrinos. *Phys. Rev.* **2007**, *D76*, 063004. [CrossRef]
104. Goldberg, H.; Perez, G.; Sarcevic, I. Mini Z' burst from relic supernova neutrinos and late neutrino masses. *J. High Energy Phys.* **2006**, *11*, 23. [CrossRef]
105. Markevitch, M.; Gonzalez, A.H.; Clowe, D.; Vikhlinin, A.; Forman, W.; Jones, C.; Murray, S.; Tucker, W. Direct Constraints on the Dark Matter Self-Interaction Cross Section from the Merging Galaxy Cluster 1E 0657-56. *ApJ* **2004**, *606*, 819. [CrossRef]
106. Boyanovsky, D.; de Vega, H.J.; Holman, R. Nonequilibrium evolution of scalar fields in FRW cosmologies. *Phys. Rev.* **1994**, *D49*, 2769. [CrossRef] [PubMed]
107. Boyanovsky, D.; de Vega, H.J.; Sanchez, N.G. CMB quadrupole suppression. I. Initial conditions of inflationary perturbations. *Phys. Rev.* **2006**, *D74*, 123006 [CrossRef]
108. Boyanovsky, D.; de Vega, H.J.; Sanchez, N.G. CMB quadrupole suppression. II. The early fast roll stage. *Phys. Rev.* **2006**, *D74*, 123007. [CrossRef]
109. Cao, C.; de Vega, H.J.; Sanchez, N.G. Quantum slow-roll and quantum fast-roll inflationary initial conditions: CMB quadrupole suppression and further effects on the low CMB multipoles. *Phys. Rev.* **2008**, *D78*, 083508. [CrossRef]
110. Destri, C.; de Vega, H.J.; Sanchez, N.G. CMB quadrupole depression produced by early fast-roll inflation: Monte Carlo Markov chains analysis of WMAP and SDSS data. *Phys. Rev.* **2008**, *D78*, 023013 [CrossRef]
111. Destri, C.; de Vega, H.J.; Sanchez, N.G. Preinflationary and inflationary fast-roll eras and their signatures in the low CMB multipoles. *Phys. Rev.* **2010**, *D81*, 063520. [CrossRef]
112. de Vega, H.J.; Sanchez, N.G. Galaxy phase-space density data exclude Bose-Einstein condensate Axion Dark Matter. *Universe* **2022**, *8*, 419. [CrossRef]
113. Special Issue keV Warm Dark Matter in Agreement with Observations in Tribute to Hector de Vega. Available online: https://www.mdpi.com/journal/universe/special_issues/kWDM (accessed on 28 May 2022).
114. de Vega, H.J.; Sanchez, N.G. Model independent analysis of dark matter points to a particle mass at the keV scale. *Mon. Not. Roy. Astron. Soc.* **2010**, *404*, 885. [CrossRef]
115. de Vega, H.J.; Sanchez, N.G. Equation of state, universal profiles, scaling and macroscopic quantum effects in warm dark matter galaxies. *Eur. Phys. J. C* **2017**, *77*, 81. [CrossRef]
116. Destri, C.; de Vega, H.J.; Sanchez, N.G. Quantum WDM fermions and gravitation determine the observed galaxy structures. *Astrop. Phys.* **2013**, *46*, 14. [CrossRef]
117. de Vega, H.J.; Sanchez, N.G. Cosmological evolution of warm dark matter fluctuations. II. Solution from small to large scales and keV sterile neutrinos. *Phys. Rev. D* **2012**, *85*, 043516–043517. [CrossRef]
118. Destri, C.; de Vega, H.J.; Sanchez, N.G. Warm dark matter primordial spectra and the onset of structure formation at redshift zz. *Phys. Rev. D* **2013**, *88*, 083512. [CrossRef]
119. de Vega, H.J.; Salucci, P.; Sanchez, N.G. Observational rotation curves and density profiles versus the Thomas–Fermi galaxy structure theory. *Mon. Not. Roy. Astron. Soc.* **2014**, *442*, 2717. [CrossRef]
120. de Vega, H.J.; Sanchez, N.G. Warm Dark Matter Galaxies with Central Supermassive Black Holes. *Universe* 2022, *8*, 154. [CrossRef]
121. Bollini, C.G.; Giambiagi, J.J. Dimensional renorinalization: The number of dimensions as a regularizing parameter. *Nuovo Cimento* **1972**, *B12*, 20. [CrossRef]
122. Bollini, C.G.; Giambiagi, J.J. Lowest order "divergent" graphs in v-dimensional space. *Phys. Lett.* **1972**, *B40*, 566. [CrossRef]
123. 't Hooft, G.; Veltman, M. Regularization and renormalization of gauge fields. *Nucl. Phys.* **1972**, *B44*, 189. [CrossRef]
124. de Vega, H.J.; Schaposnik, F.A. Dimensional renormalization. *J. Math. Phys.* **1974**, *15*, 1998. [CrossRef]