Renormalizing large-scale structure perturbation theory (or: What to do instead of the Q model)

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Three fairly independent parts of this talk:

- X Renormalization group (RG) improvement of the perturbation theory (PT) calculation of the mass power spectrum (Sabino Matarrese tomorrow)
- ✓ Connecting mass to observables: (galaxy) bias in PT.
- ? RG recovery of velocity dispersion (stream crossing) in Eulerian PT.

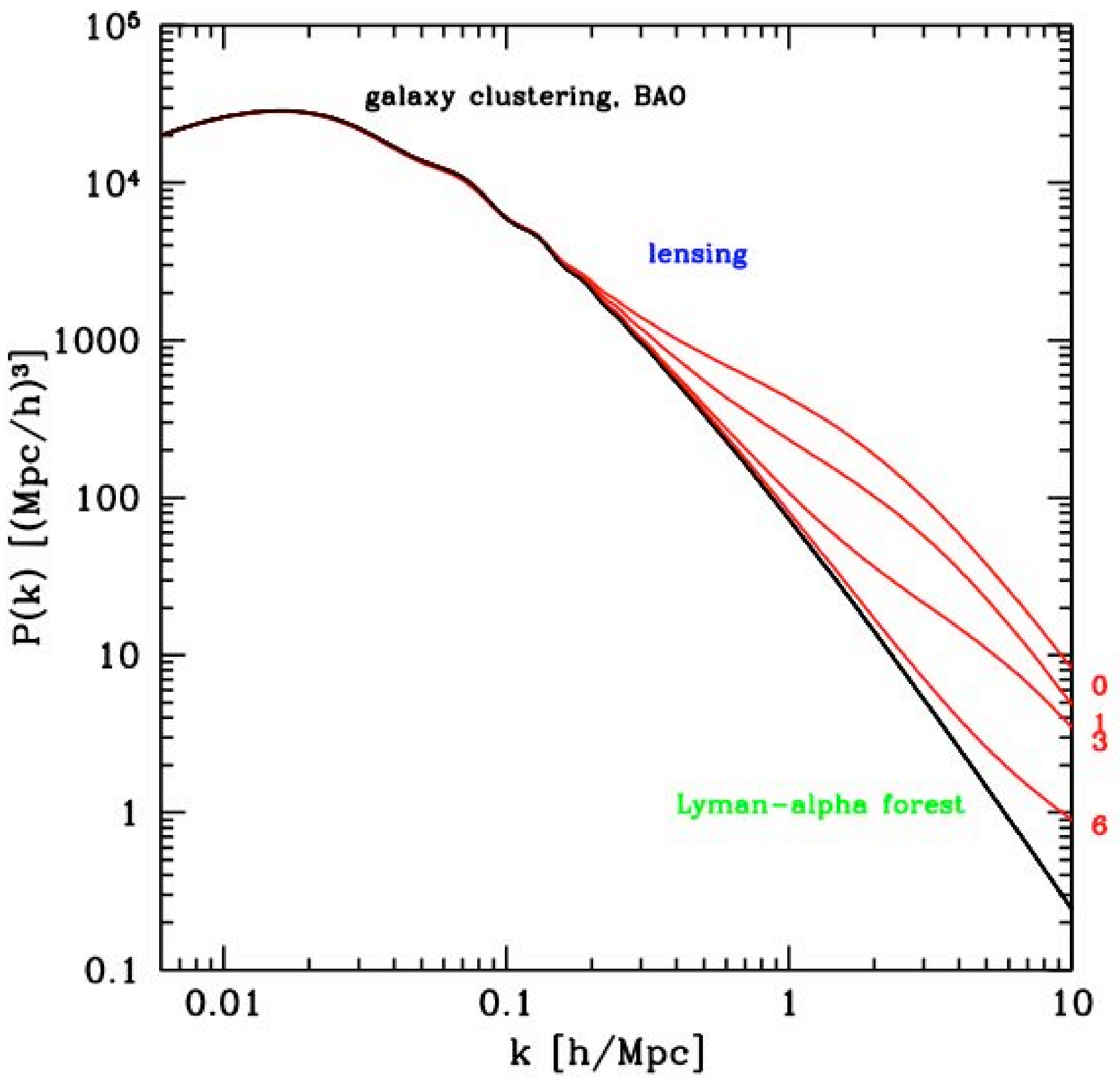
Driving motivation:

- In order to achieve the goals of future (or even current) giant large-scale structure surveys we need precise, reliable calculations of the observable statistics, e.g., the galaxy power spectrum, which include a means to marginalize over uncertainties in the model.

Motivation for precision measurements of the galaxy (or other) power spectra.

- Dark energy through baryonic acoustic oscillations (BAO) has been a big focus lately, but older reasons haven’t gone away…
- Measurement of the turnover scale gives Omega_m h.
- Neutrino masses.
- Inflation through the shape of the primordial power spectrum.
- Modified gravity, etc., etc.
- Generally fits together with other constraints (e.g., CMB) to break degeneracies.

Non-linear power spectrum



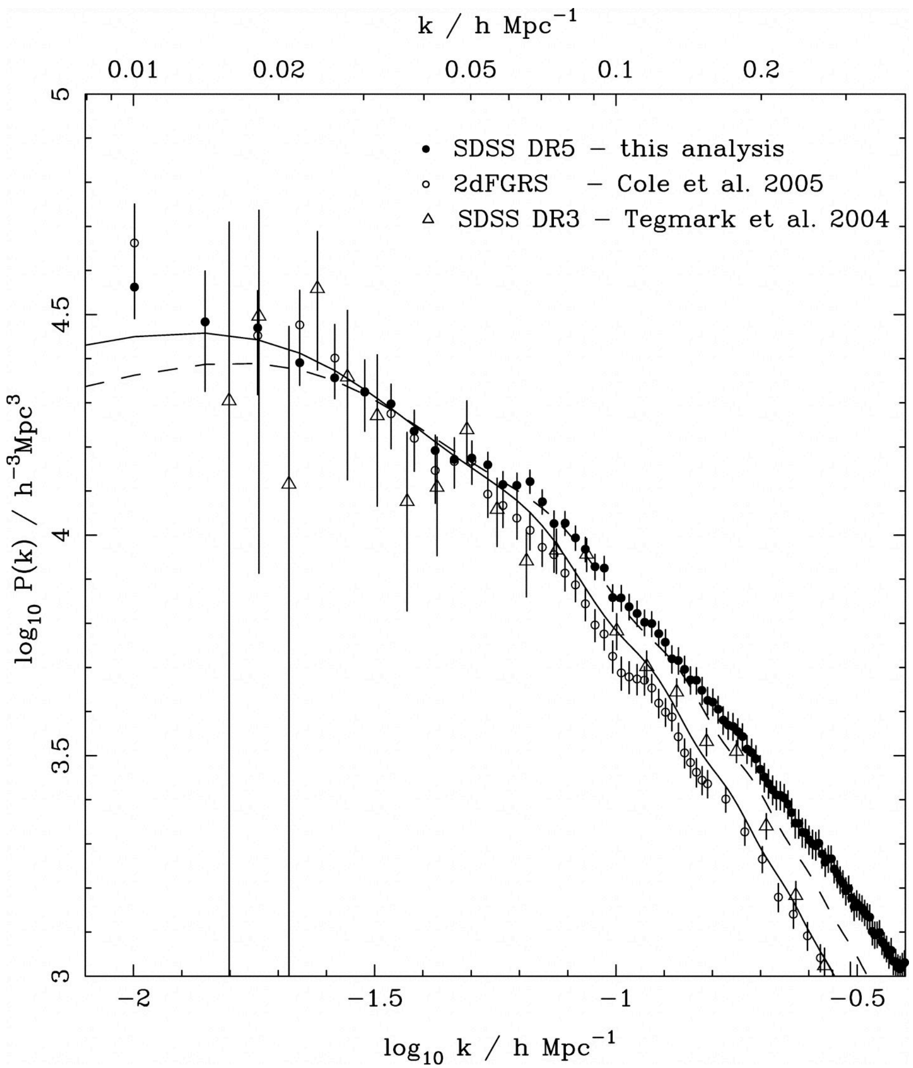
Observational Motivation:

- Galaxies/BAO
- Lya forest
- Cosmic shear
- clusters/SZ
- 21 cm(?)
- Red non-linear curves from Smith et al. simulation fits, not perfectly accurate

Conclusions

- Perturbation theory can provide a practical, elegant, immediately applicable model for galaxy bias (or other tracers of LSS). (soon, McDonald 2006)
- The "single-stream" approximation in Eulerian PT can be eliminated using RG methods, patching a hole in the foundation of PT. (soon)
- RG methods can improve the calculation of the mass power spectrum, making it accurate enough for precision cosmology. (Matarrese & Pietroni, Crocce & Scoccimarro, McDonald 2007)

SDSS-2dF P(k) comparison



- "Bias" just means generally the differences between galaxy density and mass density.
- The difference between non-linear and linear mass density is also an issue.
- On relevant scales, linear bias (galaxy density proportional to mass density) is not sufficient.

Percival et al (2007)

Two approaches to modeling galaxy clustering:

- Halo model is a bottom up (microscopic?) approach: take one fundamental thing that we know about individual galaxies - that they live in dark matter halos - and use this to predict large-scale clustering. There has been a lot of work on this and I'm not saying there's anything wrong with it.
- Perturbation theory is a top down (effective field theory?) approach: start with the fact that perturbations are small on very large scales, suggesting a Taylor series, and sweep small-scale details under the rug as much as possible. Less work so far.

For most of my career I would have said PT was a curiosity, not very relevant to interpreting real observations, so why am I working on it now?

- Criticism: It just doesn't work very well.
  - Response: Renormalization methods can fix that.
- Even to the extent that it does, it doesn't extend the range of scales accurately predicted very far.
  - The range where it helps is critical for BAO.
  - Higher precision data means a wider range where beyond-linear PT is both necessary and accurate (i.e., corrections can be very important, while still being small).
- The equations you're solving are incomplete (single-stream approximation).
  - It isn't clear that this is significant on relevant scales, but, to the extent that it is, we can fix it using RG methods.

Why not just use simulations?

- Slow and painful, to the point where no one has pushed through a complete, accurate (well-tested) mass power spectrum result, even though everyone knows it is just a matter of effort to do it.
- More importantly: galaxies, and other observable tracers of mass, can't be simulated from first principles, so it is useful to use a variety of calculational methods to get an indication of the robustness of results (plus, again, speed).

Bias of tracers (McDonald 2006)

(most of the base calculations in Heavens, Matarrese, & Verde, 1998, without the renormalization interpretation)

Naïve perturbation theory: tracer density is a Taylor series in mass density perturbation (local for now):

$$\rho_g(\delta) = \rho_0 + \rho_0' \ \delta + \frac{1}{2}\rho_0'' \ \delta^2 + \frac{1}{6}\rho_0''' \ \delta^3 + ...$$

To make sense, higher order terms should decrease in size.

Trivial warm-up: compute the mean density of galaxies:

$$\langle \rho_g \rangle = \rho_0 + \frac{1}{2}\rho_0'' \langle \delta^2 \rangle + ...$$

2nd term is ~divergent (maybe not literally infinite, depending on the power spectrum, but certainly large)

Not a problem. Eliminate the "bare" Taylor series parameter in favor of a parameter for the *observed* mean density of galaxies.

$$\bar{\rho}_g \equiv \rho_0 + \frac{1}{2}\rho_0''\sigma^2$$

$$\rho_g(\delta) = \bar{\rho}_g + \rho_0' \ \delta + \frac{1}{2}\rho_0'' \ \left(\delta^2 - \sigma^2\right) + \frac{1}{6}\rho_0''' \ \delta^3 + \ldots$$

The mean density is a trivial example, leads to nothing new.
Now move to fluctuations:

$$\delta_g\left(\delta\right) = \frac{\rho_g\left(\delta\right) - \bar{\rho}_g}{\bar{\rho}_g} = c_1\delta + \frac{1}{2}c_2\left(\delta^2 - \sigma^2\right) + \frac{1}{6}c_3\delta^3 + \ldots$$

Correlation function:

$$\xi_g\left(|\mathbf{x}_a - \mathbf{x}_b|\right) = \xi_g^{ab} = \left\langle \delta_g^a \delta_g^b \right\rangle =$$

$$c_1^2\left\langle \delta_a\delta_b\right\rangle + \tfrac{1}{3}c_1c_3\left\langle \delta_a\delta_b^3\right\rangle + \tfrac{1}{4}c_2^2\left(\left\langle \delta_a^2\delta_b^2\right\rangle - \sigma^4\right) + c_1c_2\left\langle \delta_a\delta_b^2\right\rangle + \ldots$$

(assuming 4th order terms Gaussian):

$$\xi_g^{ab} = c_1^2\xi_{ab} + c_1c_3\sigma^2\xi_{ab} + \frac{1}{2}c_2^2\xi_{ab}^2 + c_1c_2\left\langle \delta_a\delta_b^2\right\rangle + \ldots$$

Going to absorb divergent part into *observable* linear bias, but not yet because another piece comes from the cubic term.

Standard perturbation theory for gravitational collapse:

Evolution equations:

$$\frac{\partial \delta}{\partial \tau} + \nabla \cdot [(1+\delta)\,\mathbf{v}] = 0$$ Continuity

$$\frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H}\mathbf{v} + (\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\nabla\phi$$ Euler

$$\nabla^2\phi = 4\,\pi\,G\,a^2\bar{\rho}\,\delta$$ Poisson

Write density (and velocity) as a series of (ideally) increasingly small terms, δ_n is of order δ_1^n

$$\delta = \delta_1 + \delta_2 + \delta_3 + \ldots$$

Solve evolution equations iteratively

Density in standard PT

$$\delta_{\mathbf{k}} = \delta_1(\mathbf{k}) + \int \frac{d^3\mathbf{q}}{(2\pi)^3}\delta_1(\mathbf{q})\delta_1(\mathbf{k}-\mathbf{q})J_S^{(2)}(\mathbf{q},\mathbf{k}-\mathbf{q}) +$$
$$\int \frac{d^3\mathbf{q}_1}{(2\pi)^3}\frac{d^3\mathbf{q}_2}{(2\pi)^3}\delta_1(\mathbf{q}_1)\delta_1(\mathbf{q}_2)\delta_1(\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2)J_S^{(3)}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{k}-\mathbf{q}_1-\mathbf{q}_2) + \ldots$$

- Now have non-linear density field in terms of the original Gaussian fluctuations, so it is easy to evaluate statistics.

Moving to the galaxy power spectrum, and using 2nd order perturbation theory for the cubic term:

$$P_g(k) = [c_1^2 + c_1 c_3 \sigma^2 + \tfrac{68}{21} c_1 c_2 \sigma^2] P(k) +$$
$$\tfrac{1}{2} c_2^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_L(q) P_L(|\mathbf{k}-\mathbf{q}|) + 2 c_1 c_2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_L(q) P_L(|\mathbf{k}-\mathbf{q}|) J_S^{(2)}(\mathbf{q}, \mathbf{k}-\mathbf{q}) + \ldots$$

The red integral has a badly behaved part. Constant as k->0 so it looks like *shot-noise*. Absorb the constant part into a free-parameter for the *observable* shot-noise power (preserve linear bias+shot noise model on large scales):

$$N = N_0 + \frac{1}{2} c_2^2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_L^2(q)$$

$$b_1^2 = c_1^2 + c_1 c_3 \sigma^2 + \frac{68}{21} c_1 c_2 \sigma^2$$

$$\tilde{b}_2 = \frac{c_2}{b_1}$$

Final result:

$$P_g(k) = N +$$
$$b_1^2 \left[P(k) + \frac{\tilde{b}_2^2}{2} \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_L(q) \left[P_L(|\mathbf{k}-\mathbf{q}|) - P_L(q) \right] + 2\tilde{b}_2 \int \frac{d^3\mathbf{q}}{(2\pi)^3} P_L(q) P_L(|\mathbf{k}-\mathbf{q}|) J_S^{(2)}(\mathbf{q}, \mathbf{k}-\mathbf{q}) \right]$$

Started with 4-5+ parameters:

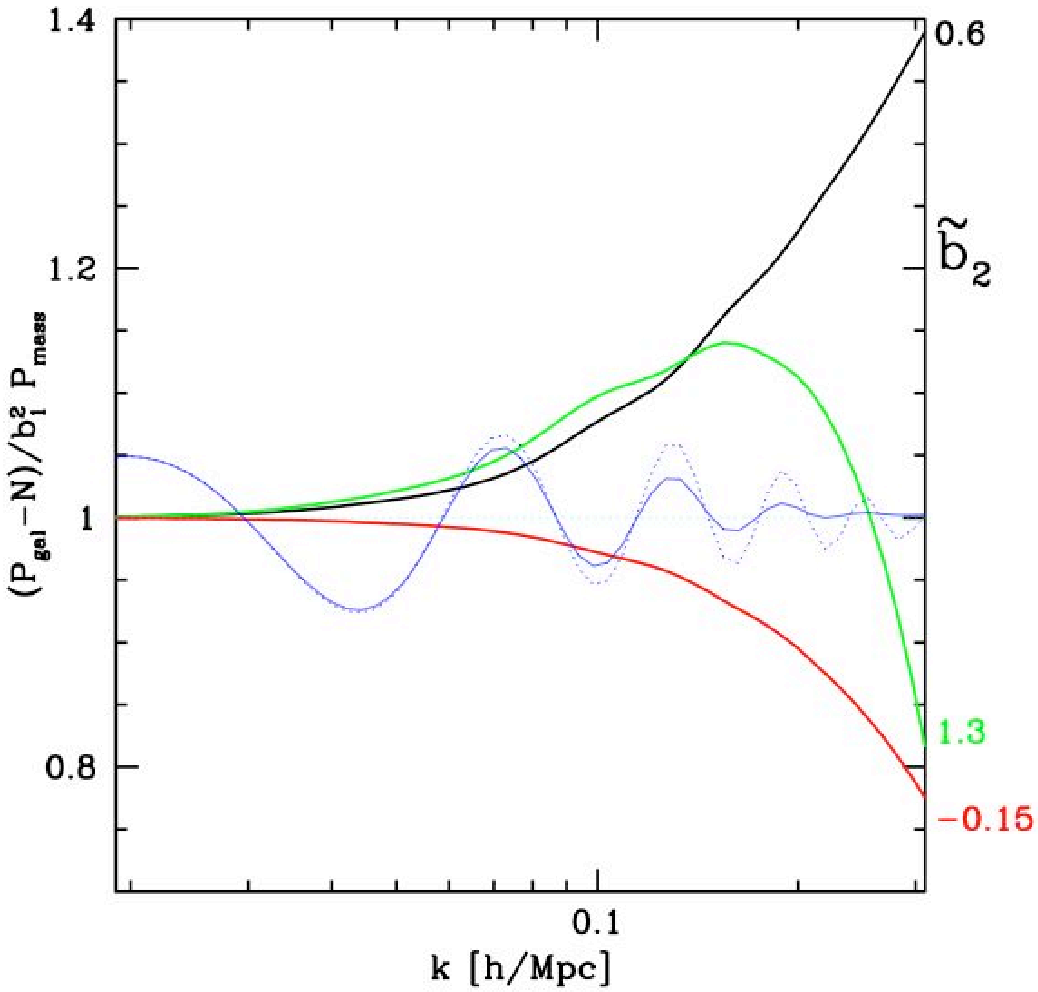
$$\rho_0,\ \rho_0',\ \rho_0'',\ \rho_0''',\ N_0,\ \text{and messy cutoffs}$$

Now have only 4, with much more cleanly separated effects:

$$\bar{\rho}_g,\ b_1,\ \tilde{b}_2,\ \text{and } N$$

(reiterate: Heavens, Matarrese, & Verde, 1998 did most of this calculation already, including recognizing the generation of "effective bias" and shot-noise)

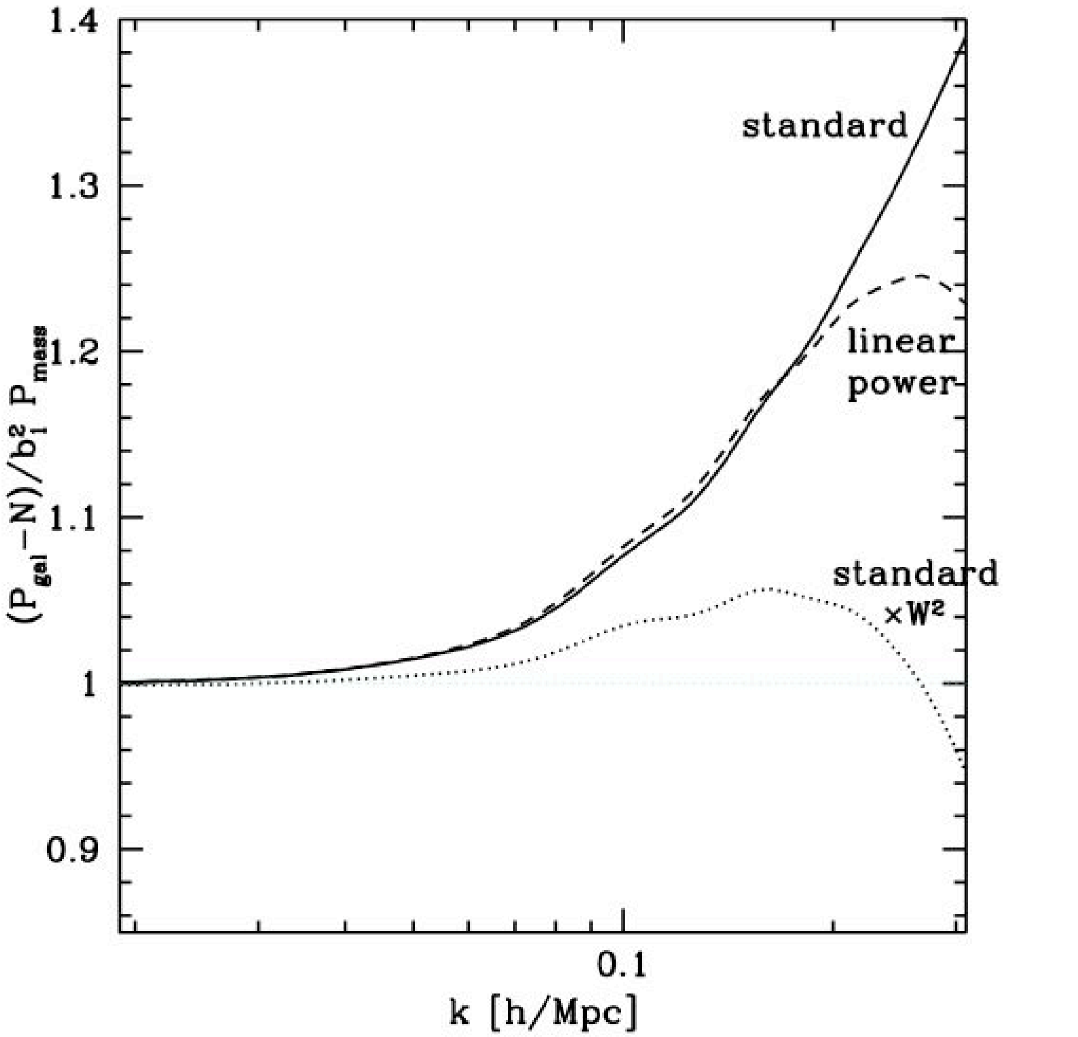
Effect of 2nd order bias in renormalized PT



Black, green, red: fundamental 2nd order bias effect, for labeled values.

Blue: BAO effect, in linear theory (dotted), and RGPT (solid)

Effect of high-k power



Standard calculation (solid) uses RG power to evaluate the bias integrals.

Dashed uses linear power.

Dotted shows the effect of 2 Mpc/h rms Gaussian smoothing (smoothing to control the Taylor series isn't a good option).

What about galaxy-mass correlation?

$$\left\langle \delta_g^a \delta^b \right\rangle = c_1 \left\langle \delta_a \delta_b \right\rangle + \frac{1}{2} c_2 \left\langle \delta_a^2 \delta_b \right\rangle + \frac{1}{6} c_3 \left\langle \delta_a^3 \delta_b \right\rangle + \ldots$$

Modified bias, consistent with the previous redefinition.
No shot-noise.

$$b_1 = c_1 + \frac{1}{2} c_3 \sigma^2 + \frac{34}{21} c_2 \sigma^2$$

Same redefinitions also work for bispectrum.
Can easily add cross-correlations between different types of galaxy.

Toward a completely general model…

- The only other variable in standard PT is the velocity divergence: $\theta = -\mathcal{H}^{-1}\nabla \cdot \mathbf{v}$
  - In linear theory there is no point in including it in the bias model because $\theta = \delta$ but this isn't true at higher orders.
  - Including $\theta - \delta$ in the Taylor series for bias adds one new free parameter (work with Arabindo Roy).
- But then we could start adding things like $\nabla^2\delta$, etc., to the model, with new free parameters.
- Also, what if the galaxy-mass relation isn't perfectly local?

Non-local model (derivative expansion)

- Galaxy density depends on mass density ~everywhere.

$$\delta_g(\mathbf{x}) = f\left[\delta(\mathbf{x}')\right]$$

- First Taylor expand in delta

$$= f[0] + \int d\mathbf{x}'\ K(|\mathbf{x}-\mathbf{x}'|)\,\delta(\mathbf{x}') + \ldots$$

- Then shift the integration variable and do a spatial expansion

$$\int d\Delta\mathbf{x}\, K(|\Delta\mathbf{x}|)\,\delta(\mathbf{x}+\Delta\mathbf{x}) = \int d\Delta\mathbf{x}\, K(|\Delta\mathbf{x}|)\left[\delta(\mathbf{x}) + \frac{d\delta}{dx_i}(\mathbf{x})\,\Delta x_i + \frac{1}{2}\frac{d^2\delta}{dx_i dx_j}(\mathbf{x})\,\Delta x_i \Delta x_j + \ldots\right]$$

$$= \delta(\mathbf{x}) \underbrace{\boxed{\int d\Delta\mathbf{x}\, K(|\Delta\mathbf{x}|)}}_{\equiv b} + \frac{d\delta(\mathbf{x})}{dx_i} \underbrace{\int d\Delta\mathbf{x}\, K(|\Delta\mathbf{x}|)\,\Delta x_i}_{0} + \frac{1}{2}\frac{d^2\delta(\mathbf{x})}{dx_i dx_j} \underbrace{\boxed{\int d\Delta\mathbf{x}\, K(|\Delta\mathbf{x}|)\,\Delta x_i \Delta x_j}}_{= b\ R^2\ \delta_{ij}\ \mathcal{O}(1)} + \ldots$$

$$\delta_g(\mathbf{x}) = b\left[\delta(\mathbf{x}) + \frac{\mathcal{O}(1)}{2} R^2\ \nabla^2 \delta(\mathbf{x})\right] + \ldots$$

Result is an expansion in the non-locality scale R over the scale you're observing, 1/k

$$\delta_g(\mathbf{k}) = b\left[1 - \frac{\mathcal{O}(1)}{2} R^2 k^2\right] \delta(\mathbf{k}) + \ldots$$

$$P_g(k) = b^2\left[1 - \tilde{b}_{k^2} R^2 k^2\right] P(k) + \ldots$$

- What looked like it might be a nightmare of new parameters turns out to be remarkably simple!
- For a reasonable R~1 Mpc/h, correction is ~4% at k=0.2 h/Mpc.
- R depends on real physics of galaxy formation - should study with simulations/semi-analytic models.

Future directions

- Cosmological parameter estimation.
- Comparison to simulations.
- Finish generalizing the model.
- Redshift-space (easy if no velocity bias)
- Dynamic model?
  - Make a copy of the mass density evolution equations to describe galaxy density evolution, and add something like
    $$\dot{\rho}_g = f(\rho)$$
    to describe galaxy formation. This kind of this has been considered before, but now we have new tools (RG) to deal with it.

Re-introducing velocity dispersion in Eulerian PT

- The “single-stream” (hydrodynamic) approximation appears to be a fundamental problem with PT, i.e., the equations we’re solving simply aren’t complete, so even if the result converges, we can’t be confident that it is correct.
- I’m going to solve this problem, which one might argue is intrinsically interesting beyond the relevance for practical uses of PT to describe observations.

Digression: Renormalization group method

- The method I use was introduced for solving differential equations by Chen, Goldenfeld, & Oono (1994). (Kunihiro and Tsumura description makes more sense to me.)
- Could be generally useful.
- Easiest to explain through a very simple example, where delta at least starts small.

$$\dot{\delta} = \delta + \delta^2$$

- As in cosmological PT, solve iteratively

$$\delta = \delta_1 + \delta_2 + \delta_3 + \ldots$$

$$\dot{\delta} = \delta + \delta^2$$

First order solution: $\delta_1 = g_1 e^t$

Equation for delta_2: $\dot{\delta}_2 = \delta_2 + g_1^2 e^{2t}$

Solution for delta_2: $\delta_2 = g_2 e^t + g_1^2 e^{2t}$

Note that I've kept the homogeneous solution, while in cosmological PT g_2 is assumed to be zero.

In this approach, the 2nd order solution inevitably grows to be larger than the 1st, invalidating the PT (when this happens depends on k in the cosmological case).

Note that solving two differential equations has produced two parameters of the solution, when only one is needed to satisfy the boundary conditions -> ambiguity.

$$\delta_2 = g_2 e^t + g_1^2 e^{2t}$$

Note that g_2 can always be chosen to make delta_2=0 at one particular time, $t_\star$

This leads to the full (1st+2nd order) solution

$$\delta \simeq g_1 e^t + g_1^2 e^t (e^t - e^{t_\star}).$$

Perturbation theory will be valid near $t_\star$, but break if you go very far away. g_1 can be fixed to satisfy the boundary conditions, but note that it's value will depend on $t_\star$ i.e,

$$g_1 = g_1(t_\star)$$

The RG method is to impose the fact that the full solution, delta, should not depend on $t_\star$, producing a differential equation for $g_1(t_\star)$

$$\delta \simeq g_1 e^t + g_1^2 e^t (e^t - e^{t_\star}).$$

$$\left.\frac{d\delta(t)}{dt_\star}\right|_{t_\star = t} = 0 = \frac{dg_1(t)}{dt} e^t - g_1^2 e^{2t}$$

gives $\quad \dfrac{dg_1(t)}{dt} = g_1^2 e^t$

Solution: $\quad g_1(t) = \dfrac{c}{1 - c\ e^t}$

Final solution: $\quad \delta(t) = \dfrac{c\ e^t}{1 - c\ e^t}$

This is the exact solution to the original difeq! (lucky)

Intuitively, you can think of this calculation as stepping forward in time slightly using the perturbative solution valid near the present time, then taking the result and using it as the initial condition for a perturbative solution around the new time, followed by another step, etc…

Re-introducing velocity dispersion in Eulerian PT

- The “single-stream” (hydrodynamic) approximation appears to be a fundamental problem with PT, i.e., the equations we’re solving simply aren’t complete, so even if the result converges, we can’t be confident that it is correct.
- I’m going to solve this problem, which one might argue is intrinsically interesting beyond the relevance for practical uses of PT to describe observations.

- In Eulerian PT, why not just add equations for the velocity dispersion and higher moments? It's actually not hard to write down the equations…
- The exact description of CDM is the Vlasov and Poisson equations:

$$\frac{\partial f}{\partial \tau} + \frac{1}{a\ m}\ \mathbf{p} \cdot \nabla f - a\ m\ \nabla\phi \cdot \nabla_{\mathbf{p}} f = 0$$

$$\nabla^2 \phi = 4\ \pi\ G\ a^2 \bar{\rho}\ \delta$$

Where $f(\mathbf{x}, \mathbf{p}, \tau)$ is the particle phase-space distribution function.

- The 0th moment of the Vlasov equation with respect to momentum gives the continuity equation (Peebles 1980):

$$\frac{\partial \delta}{\partial \tau} + \partial_i \left[(1+\delta)\, v^i \right] = 0$$

Where density and velocity are moments of f with respect to p.

- The 1st moment gives something like the Euler equation:

$$\frac{\partial v^i}{\partial \tau} + \mathcal{H} v^i + v^j \partial_j v^i = -\partial_i \phi - \frac{\partial_j \left[(1+\delta)\, \sigma^{ij} \right]}{1+\delta}.$$

Where $\sigma^{ij}(\mathbf{x}, \tau) \equiv \left\langle \delta v^i \ \delta v^j \right\rangle$ is usually set to zero.

- Now need an evolution equation for sigma_ij, but this is just the next moment of the Vlasov equation:

$$\frac{\partial \sigma^{ij}}{\partial \tau} + 2\,\mathcal{H}\,\sigma^{ij} + v^k\,\partial_k \sigma^{ij} + \sigma^{jk}\,\partial_k v^i + \sigma^{ik}\,\partial_k v^j = -\frac{\partial_k\left[(1+\delta)\,q^{ijk}\right]}{1+\delta}$$

with $q^{ijk}(\mathbf{x},\tau) \equiv \left\langle \delta v^i\ \delta v^j\ \delta v^k \right\rangle$

- You might think we could just assume q=0, repeat the usual perturbation theory calculations including this equation, and obtain interesting new results… but it doesn't work.

- Sigma_ij can have a homogeneous, zero order, component $\bar{\sigma}_0(\tau) = \bar{\sigma}_0^{11} = \bar{\sigma}_0^{22} = \bar{\sigma}_0^{33}$
- Which solves $$\frac{\partial \bar{\sigma}_0}{\partial \tau} + 2\ \mathcal{H}\ \bar{\sigma}_0 = 0$$
- Assume EdS universe so $a = \left(\frac{\tau H_0}{2}\right)^2, \quad \mathcal{H} = \frac{2}{\tau}$

$$\bar{\sigma}_0(\tau) = A\tau^{-4}$$

- For CDM this starts very small and gets rapidly smaller.

- Linear equations, Fourier space

$$
\begin{aligned}
\frac{\partial \delta_1}{\partial \tau} + \theta_1 &= 0 \\
\frac{\partial \theta_1}{\partial \tau} + \frac{2}{\tau}\theta_1 + \frac{6}{\tau^2}\delta_1 &= -\pi_1 + k^2 \bar{\sigma}_0 \delta_1 \\
\frac{\partial \pi_1}{\partial \tau} + \frac{4}{\tau}\, \pi_1 - 2k^2 \bar{\sigma}_0\, \theta_1 &= 0
\end{aligned}
$$

$$\theta = \nabla \cdot \mathbf{v} \qquad \pi = \partial_i \partial_j \sigma^{ij}$$

- This is the end of the story in standard PT. The only source for dispersion is the very tiny zero order dispersion.
- Vorticity follows a similar story.

Very vexing problem!

- Velocity dispersion, i.e., stream-crossing, is obviously ubiquitous in the real Universe.
- Conventional wisdom, probably motivated by Zel'dovich approximation-type thinking, is that stream-crossing is fundamentally non-perturbative, i.e., the situation is hopeless (Afshordi 2007).
- Let's push on with this calculation anyway.

- Can’t solve the 1st order equations exactly, but can solve iteratively treating sigma_0 k^2 terms as perturbations. Find, after some transients have died:

$$\delta_1 = \delta_1(k \to 0)\left(1 - \frac{k^2 A}{5\tau_i^2}\right)$$

Makes sense: frozen-out Jean’s smoothing.

$$\mathbf{v}_1 = -i\mathcal{H}\frac{\mathbf{k}}{k^2}\delta_1$$

Usual linear theory.

$$\sigma_1^{ij} = 2\bar{\sigma}_0\frac{k_i k_j}{k^2}\delta_1$$

New thing.
Still no reason to think these terms aren’t ridiculously small.

- Finally, 2nd order equation for sigma_ij:

$$\frac{\partial \sigma_2^{ij}}{\partial \tau} + 2\ \mathcal{H}\ \sigma_2^{ij} + \boxed{v_1^k\ \partial_k \sigma_1^{ij} + \sigma_1^{jk}\ \partial_k v_1^i + \sigma_1^{ik}\ \partial_k v_1^j} + \bar{\sigma}_0\ \partial_j v_2^i + \bar{\sigma}_0\ \partial_i v_2^j = 0$$

- Interested in mean (zero mode) which will renormalize the homogeneous zero order dispersion,

$$\frac{\partial \bar{\sigma}_2}{\partial \tau} + 2\ \mathcal{H}\ \bar{\sigma}_2 + \frac{1}{3} \left\langle v_1^k\ \partial_k \sigma_1^{ii} + 2\sigma_1^{ik}\ \partial_k v_1^i \right\rangle = 0$$

where $\bar{\sigma}_2 \equiv \left\langle \sigma_2^{11} \right\rangle = \left\langle \sigma_2^{22} \right\rangle = \left\langle \sigma_2^{33} \right\rangle = \sigma_2^{ii}/3$

- Evaluating using the 1st order solutions gives:

$$\frac{\partial \bar{\sigma}_2}{\partial \tau} + 2\ \mathcal{H}\ \bar{\sigma}_2 = \frac{2}{3} \mathcal{H} \bar{\sigma}_0 \left\langle \delta_1^2 \right\rangle$$

- Simple solution using known delta_1

$$\bar{\sigma}_2 = \frac{1}{3}\bar{\sigma}_0 \left\langle \delta_1^2 \right\rangle + \frac{c}{\tau^4}$$

- Key point is that once the total density variance is >1 the perturbative expansion breaks down. The 2nd order term is growing rapidly relative to the 0th. This is where the renormalization group enters.
- We can always chose the superfluous parameter c to make the 2nd order term zero at one particular time $\tau_\star$

- Full solution

$$\bar{\sigma}(\tau) = A_\star \tau^{-4} + \frac{1}{3}\left[A_\star \tau^{-4}\left\langle \delta_1^2 \right\rangle - A_\star \left\langle \delta_1^2\left(\tau_\star\right)\right\rangle \tau^{-4}\right]$$

- Obtain an RG equation for $A_\star \equiv A(\tau_\star)$

$$\left.\frac{d\bar{\sigma}}{d\tau_\star}\right|_{\tau_\star=\tau} = 0 = \frac{dA_\star}{d\tau}\tau^{-4} - \frac{1}{3}A_\star\tau^{-4}\frac{d\left\langle \delta_1^2 \right\rangle}{d\tau}$$

$$\frac{dA_\star}{d\tau} = \frac{1}{3}A_\star\frac{d\left\langle \delta_1^2 \right\rangle}{d\tau} \qquad ?\ \frac{A_\star}{A_i} = \exp\left(\sigma^2\right)\ ?$$

- This equation is telling us how to feed the 2nd order velocity dispersion back into the 0th order.

- Recall that delta_1 is smoothed by the velocity dispersion itself, with the approximate solution

$$\delta_1 = \delta_1(k \to 0)\left(1 - \frac{k^2 A}{5\tau_i^2}\right)$$

- This was a small-k expansion, and represents Jeans-like smoothing by velocity dispersion. Assuming the smoothing is a Gaussian exp[-(k R_F)^2/2] gives

$$R_F = \left(\frac{2A}{5\tau^2}\right)^{\frac{1}{2}}$$

Where I've assumed the smoothing takes place effectively instantaneously.

- These approximations make this basically an order-of-magnitude calculation.

- With Gaussian smoothing and a power law power spectrum $\Delta^2(k,\tau) = \Delta_p^2(\tau)(k/k_p)^{3+n}$

$$\left\langle \delta_1^2 \right\rangle = \frac{\Delta_p^2(\tau)}{2} \frac{\Gamma\left[(3+n)/2\right]}{\left[k_p R_F\right]^{3+n}} \qquad (n > -3)$$

- We can now solve the RG equation

$$\frac{dA}{d\tau} = \frac{1}{3} A \frac{d\left\langle \delta_1^2 \right\rangle}{d\tau} = \mathrm{const}\, A^{-\frac{3+n}{2}} \tau^{6+n}$$

- Initial conditions are forgotten, leaving

$$A=\left[\frac{3+n}{6}\Delta_p^2\left(\tau\right)\frac{\Gamma\left(\left(3+n\right)/2\right)}{\left[k_p\sqrt{2/5}\;\tau^{-1}\right]^{3+n}}\right]^{\frac{2}{3+n}}$$

where $\bar{\sigma}_0(\tau) = A\tau^{-4}$

- Easy to understand when re-written

$$\frac{6}{3+n}=\left\langle\delta_1^2\left[R_F\left(A\right)\right]\right\rangle$$

Final result

- Remember, just an order of magnitude calculation, but the result is simply that the filtering scale grows to keep the rms fluctuation level of order 1, with the exact coefficient dependent on the slope of the power spectrum.

$$\frac{6}{3+n} = \left\langle \delta_1^2 \left[R_F \left(A \right) \right] \right\rangle$$

- All of this could be done numerically to obtain results accurate in detail.

- The linear power is truncated, but higher order corrections can regenerate it.
- The effect appears somewhat similar to Crocce & Scoccimarro’s propagator renormalization, although of apparently different origin.
- Work remains to be done to give this practical value!