The Inflationary Perturbation Spectrum: Numerical and Analytical Calculations

Andreas Heinen[1]

[1]Theoretical Physics III
University of Dortmund
Andreas.Heinen@uni-dortmund.de

DO-TH-04/14, LA-UR-04-8611

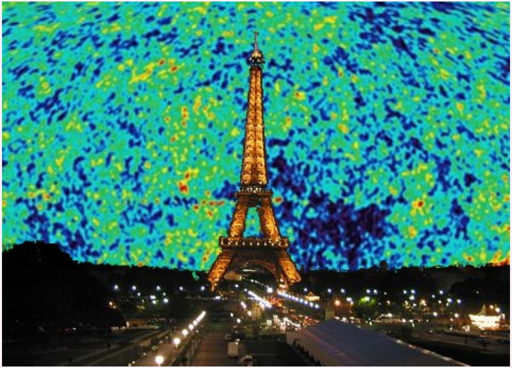
9/10 Dec 2004, WMAP and the Early Universe, Paris

in Collaboration with Salman Habib, Katrin Heitmann, Gerard Jungman (Los Alamos Nat Lab) and Carmen Molina-Paris (U of Leeds)

[1] S.H., K.H., G.J., C.M.-P., Phys. Rev. Lett. **89**, 281301 (2002), [2] S.H., A.H., K.H., G.J., C.M.-P., Phys. Rev. **D70**, 083507 (2004), [3] S.H., A.H., K.H., G.J., *to be published*, astro-ph/0412xxx

Outline

Inflation and its imprints in the microwave sky

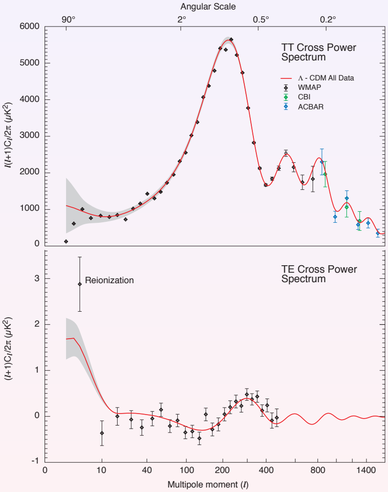


Quantum fluctuations in the very early universe are blown up to astrophysical size (large scale structure)

1. **Inflation** provides a natural mechanism for the generation of perturbations.
2. One can constrain inflationary models from **cosmic microwave background** (CMB) measurements (WMAP, PLANCK,. . .) and the **large scale structure** (SDSS, 2dFGRS,. . .)
3. Inflation is testable!
   H. V. Peiris *et al.*, Astrophys. J. Suppl. **148**, 213 (2003)

Components of the Cosmic Microwave Background

D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003)



1. Primordial perturbation spectrum (very early universe)
2. Mixing of matter and radiation in the epoch of recombination, "late" universe → sound waves, anisotropies, ...

We can only measure the combination

$$C_\ell = 4\pi \int \frac{dk}{k} P(k) \Delta_\ell^2(k) \qquad (1)$$

$P(k)$: primordial power spectrum (**this talk**)
$\Delta_\ell^2(k)$: transfer function → today (CMBFAST,...)

Summary of the main theoretical ingredients

Linearized fluctuations in the metric and a background field – single field inflation

V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992)

$$ds^2 = g_{ab} dx^a dx^b = a^2(\eta) \left[-d\eta^2 + \gamma_{ij} dx^i dx^j \right] \quad (2)$$

density perturbations $\longrightarrow$ scalar

metric perturbations (gravity waves) $\longrightarrow$ tensor

Characterizing quantity: Gauge invariant variable u satisfying the dynamical equation (for scalar perturbations)

$$u_k'' + \left(k^2 - \frac{z''}{z} \right) u_k = 0 \; ; \; z = \frac{a\phi'}{h} \; ; \; h = \frac{a'}{a} \quad (3)$$

Background equations for the inflaton field

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + V'(\phi(t)) = 0 \; ; \; H^2(t) = \frac{1}{3} \left[\frac{1}{2} \dot{\phi}^2(t) + V(\phi) \right] \; . \quad (4)$$

The power spectra of cosmological perturbations

Two-point correlations

For Gaussian fluctuations the two-point correlation function encodes all the information needed!

$$\langle 0|\, u(\eta,\mathbf{x}), u(\eta,\mathbf{y})\,|0\rangle = \int_0^\infty \frac{dk}{k}\frac{\sin kr}{kr}P_u(\eta,k) \quad ; \quad r = |\mathbf{x}-\mathbf{y}| \qquad (5)$$

Cosmological perturbations

The **growing modes** (on scales much larger than the Hubble length) build up the perturbation spectrum.
Power spectrum for density perturbations (ζ) and curvature perturbations (g):

$$P_\zeta(k) = P_S(k) \overset{k\eta\to 0^-}{=} \frac{k^3}{2\pi^2}\left|\frac{u_k(\eta)}{z(\eta)}\right|^2 \qquad (6)$$

$$\frac{1}{8}P_g(k) = P_T(k) \overset{k\eta\to 0^-}{=} \frac{k^3}{2\pi^2}\left|\frac{v_k(\eta)}{a(\eta)}\right|^2 \qquad (7)$$

Summary of the characterizing quantities

Power spectra

$$P_S(k) \overset{k\eta\to 0^-}{=} \frac{k^3}{2\pi^2}\left|\frac{u_k(\eta)}{z(\eta)}\right|^2 \quad (8)$$

$$P_T(k) \overset{k\eta\to 0^-}{=} \frac{k^3}{2\pi^2}\left|\frac{v_k(\eta)}{a(\eta)}\right|^2 \quad (9)$$

Ratio of tensor to scalar perturbations

$$R(k) = \frac{8P_T(k)}{P_S(k)} \quad (10)$$

Spectral indices

$$n_S(k) = 1 + \frac{d\ln P_S(k)}{d\ln k} \quad (11)$$

$$n_T(k) = \frac{d\ln P_T(k)}{d\ln k} \quad (12)$$

Running of the spectral indices

$$\alpha_S(k) = \frac{d\ln n_S(k)}{d\ln k} \quad (13)$$

$$\alpha_T(k) = \frac{d\ln n_T(k)}{d\ln k} \quad (14)$$

Precision calculation of $P(k)$?

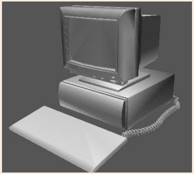
Important requirements

1. The errors in the calculation of the **primordial perturbation spectra** should be much smaller than the "cosmic variance".
2. The errors should be comparable to (or smaller) the present $\sim 0.1\%$ errors in the calculation of the transfer functions.

   CMBFAST: `http://www.cmbfast.org`; U. Seljak and M. Zaldarriaga, Astrophys. J. **469**, 437 (1996); U. Seljak, N. Sugiyama, M. White and M. Zaldarriaga, Phys. Rev. D **68**, 083507 (2003)

Numerical calculations

- The only method reliable in the general case
- Numerically too expensive for model testing/parameter studies
- Only useful for spot-checking



The standard slow-roll expansion (and problems)

Features

- Expansion in terms of slow-roll parameters

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{\dot{\phi}^2}{H^2} \quad ; \quad \delta_n = \frac{1}{H^n \dot{\phi}} \frac{d^{n+1}\phi}{dt^{n+1}} \tag{15}$$

- E.g. scalar spectral index to first order: $n_S = 1 - 4\epsilon - 2\delta_1$
- Numerically cheap → quick tests → Precision? Error control?
- Good, if the dynamics is completely friction dominated . . .
- Derivative expansions in terms of V, V'', V''' etc. can be dangerous (convergence?)
- Obtaining higher orders? (Bessel function approach needs a constant inhomogeneity)

Outline

The uniform approximation – general formalism

Liouville-Green (LG) and Wentzel-Kramers-Brillouin (WKB) approximations

Green (1837), Liouville (1837), R.E. Langer (1931, 1932, 1935, . . . 1949)

Cosmological perturbations: J. Martin and D. J. Schwarz, Phys. Rev. D **67**, 083512 (2003); P. Hunt and S. Sarkar, arXiv:astro-ph/0408138; R. Casadio, F. Finelli, M. Luzzi and G. Venturi, arXiv:gr-qc/0410092

Uniform approximations

- Better treatment of the turning points, "uniformization" of the LG approximation and error control

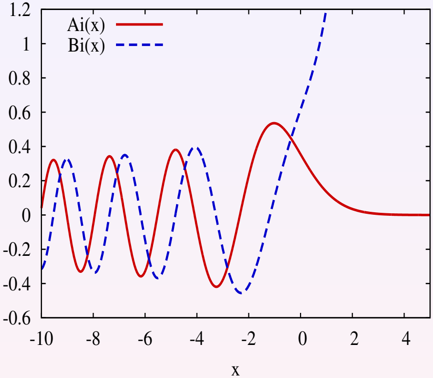
R.E. Langer (1931, 1932, 1935, . . . 1949), F.W.J. Olver, Philos. Trans. R. Soc. A **247**, 307 (1954); *Asymptotics and Special Functions*, (AKP Classics, Wellesley, MA 1997)

Cosmological perturbations: S. Habib, K. Heitmann, G. Jungman, C. Molina-Paris, Phys. Rev. Lett. **89**, 281301 (2002); S.H., A.H., K.H., G.J., C. M.-P., Phys. Rev. **D70**, 083507 (2004); S.H., A.H., K.H., G.J.,*to be published*, astro-ph/0412xxx

Features of the uniform solutions

- Solutions in terms of Airy functions
- General error control theory → error bounds for u_k, $P(k)$, $n(k)$ etc.
- No matching at the turning point → no loss of precision
- Arbitrary high orders possible (typically not needed)

Ai- and Bi-functions:



The fundamental quantity is the turning point $k^2 = \nu^2(\bar{\eta})/\bar{\eta}^2$. Solutions "$>$" with $\eta > \bar{\eta}$ (relevant for the perturbation spectrum) and "$<$" with $\eta < \bar{\eta}$.

Outline

Leading order expressions in terms of Airy functions

Unnormalized solutions (scalar perturbations)

$$u^{(1)}_{k,\lessgtr}(\eta) \propto \mathbf{Ai}[f_{\lessgtr}(k,\eta)] \tag{16}$$

$$u^{(2)}_{k,\lessgtr}(\eta) \propto \mathbf{Bi}[f_{\lessgtr}(k,\eta)] \tag{17}$$

with

$$f_{\lessgtr}(k,\eta) = \mp\left\{\pm\frac{3}{2}\int_{\eta}^{\bar{\eta}_S} d\eta' \left[\mp g_S(k,\eta')\right]^{\frac{1}{2}}\right\}^{\frac{2}{3}} \tag{18}$$

$$g_S(k,\eta) = \frac{\nu_S^2(\eta)}{\eta^2} - k^2 \ , \tag{19}$$

$$\nu_S^2 = (z''/z)\eta^2 + 1/4 \quad ; \quad \nu_T^2 = (a''/a)\eta^2 + 1/4 \ . \tag{20}$$

Note: The term $-\frac{1}{4\eta^2}$ is extracted to make the uniform approximation convergent.

The solutions have error bounds $\longrightarrow$ error bands

Simple estimate for the error bound of $u_{\lessgtr}$

...from a local expansion around the turning point:

$$|\epsilon_{k,1,\lessgtr}| \leq 2\sqrt{2}\left(\frac{1}{6\bar{\nu}_S(k)} + \frac{\lambda}{72\bar{\nu}_S^2(k)} + \cdots\right) . \quad (21)$$

$\lambda \simeq 1.04$

S. Habib, A.H., K. Heitmann, G. Jungman, *to be published*, astro-ph/0412xxx

For the full error control function see:

S. Habib, A.H., K. Heitmann, G. Jungman, C. Molina-Paris, Phys. Rev. **D70**, 083507 (2004)

The leading order power spectrum and spectral index

Leading order power spectra in the Liouville-Green regime

$$P_{1,S}(k) = \lim_{k\eta\to 0^-} \frac{k^3}{4\pi^2}\frac{1}{|z(\eta)|^2}\frac{-\eta}{\nu_S(\eta)} \exp\left\{\frac{4}{3}\left[f_>(k,\eta)\right]^{3/2}\right\} \times(1+\epsilon^P_{k,1,S}) \quad (22)$$

$$\epsilon^P_{k,1,S} = 2\epsilon_{k,1,S,\lessgtr} + \epsilon^2_{k,1,S,\lessgtr} \quad (23)$$

Leading order spectral indices with inverse square root singularities

$$n_{1,S}(k) = 4 - 2k^2 \lim_{k\eta\to 0^-} \int_{\bar{\eta}_S}^{\eta} \frac{d\eta'}{\sqrt{g_S(k,\eta')}} + \epsilon^n_{k,1,S}. \quad (24)$$

$$\epsilon^n_{k,1,S} = \frac{d\ln \epsilon^P_{k,1,S}}{d\ln k} \quad (25)$$

Analog in the tensor case ($\nu_S \to \nu_T$ and $z \to a$).

Analytical solutions for power-law inflation – constant ν

Some exponential potentials are analytically solvable

$$V(\phi) = V_0 \exp\left(\sqrt{\frac{2}{p}}\,\phi\right) \quad ; \quad a(t) \propto t^p \quad ; \quad H(t) = pt^{-1} \tag{26}$$

see e.g., E. Lidsey et al. Rev. Mod. Phys. **69**, 373 (1997)

The exact power spectrum is proportional to a Γ-function

$$P_S^{\mathrm{ex}}(k) = \frac{2^{2\nu_S-2}}{2\pi^3}\Gamma^2(\nu_S)\left(\frac{H}{a\dot{\phi}}\right)^2(-k\eta)^{1-2\nu_S}k^2 \tag{27}$$

$$\nu_S = \nu_T = \frac{3}{2} + \frac{1}{p-1} \tag{28}$$

$$n_S = 4 - 2\nu_S \quad ; \quad n_T = n_S - 1 \tag{29}$$

1. No running of the spectral index
2. Harrison-Zeldovich $n_S = 1$ corresponds to $p = \infty$

LO and NLO uniform power spectra for power-law inflation

$$\textbf{LO}: \quad P_S^{(1)}(k) = \frac{2^{2\nu_S-2}}{\pi^2} e^{-2\nu_S} \nu_S^{2\nu_S-1} \left(\frac{H}{a\dot{\phi}}\right)^2 (-k\eta)^{1-2\nu_S} k^2 \quad (30)$$

$$n_S^{(1)}(k) = 4 - 2\nu_S \quad (31)$$

$$\textbf{NLO}: \quad P_S^{(2)}(k) = P_S^{(1)}(k)\left(1 + \frac{1}{6\nu_S}\right) \quad (32)$$

$$n_S^{(2)}(k) = n_S^{(1)}(k) \quad (33)$$

Important notes about the uniform approximation

1. The spectral indices are already exact in leading order!
2. The Ratio R of tensor to scalar perturbations is exact!
3. Order by order expanded the uniform approximation reproduces the Γ-function (Stirling formula):

$$\Gamma^2(\nu) \rightarrow \left(\underbrace{1}_{\textbf{LO}} + \underbrace{\frac{1}{6\nu}}_{\textbf{NLO}} + \underbrace{\frac{1}{72\nu^2}}_{\textbf{NNLO}} + \cdots \right) =: \gamma(\nu) \quad (34)$$

Outline

Do I have to improve the leading order approximation?

Question: How do I know if I have to improve my approximation?

$\Downarrow$

Answer: There is an error bound! If the error bound is much bigger than your desired accuracy you have to improve something ($\rightarrow$ next order)

$\Downarrow$ *Too lazy to calculate the full next-to-leading order?* $\Downarrow$

Idea: Utilize the results from the power-law case in order to improve the spectra of the leading order very simply!

Definition of improved power spectra

Define improved leading order power spectra

$$\tilde{P}_{1,S}(k) = P_{1,S}(k)\gamma[\bar{\nu}_S(k)] \quad ; \tilde{P}_{1,T}(k) = P_{1,T}(k)\gamma[\bar{\nu}_T(k)] \tag{35}$$

$$\tilde{n}_{1,S}(k) = n_{1,S}(k) + \frac{d \ln \gamma[\bar{\nu}_S(k)]}{d \ln k} \tag{36}$$

Part of the Γ-function with ν at the turning point $\longrightarrow$ all local contributions from the non-local integrals of the higher orders

S. Habib, A.H., K. Heitmann, G. Jungman, *to be published*, astro-ph/0412xxx

Notes about the size of the correction

1. Bigger correction to the amplitude of the power spectrum

$$\gamma[\bar{\nu}_S(k)] \approx \epsilon^P_{k,1,S} \tag{37}$$

2. Small correction to the spectral index

$$\frac{d \ln \gamma[\bar{\nu}_S(k)]}{d \ln k} \ll \epsilon^n_{k,1,S} \tag{38}$$

Errors to be expected in the uniform approximation

Quantity	**LO**	**improved LO**	**NLO**
$P(k)$	$\sim 10\%$	$< 1\%$	$\sim 0.5\%$
$n(k)$	$< 0.1\%$	$< 0.1\%$	$\ll 0.1\%$
$R(k)$	$\sim 1\%$	$\sim 0.1\%$	$< 0.1\%$

Outline

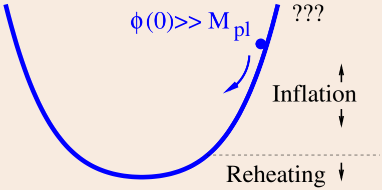
1. General Introduction and calculation of primordial calculations
2. The uniform approximation
   - General formalism
   - Leading order expressions with error bounds
   - Improvements and comments on errors
3. Results
   - Chaotic inflation
   - Phase transition model
4. Summary & Conclusions

Chaotic inflation

The models

Chaotic inflation with $V(\phi) \sim \phi^n$

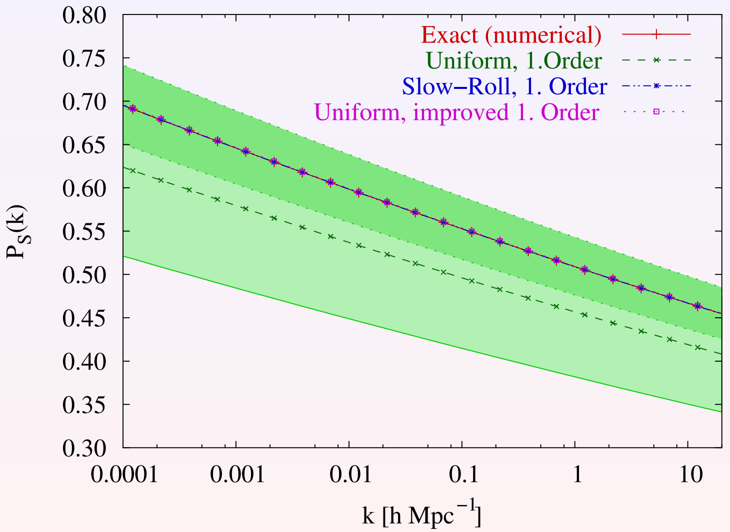
- ϕ^2 possible
- ϕ^4 under strong pressure from observations
- ϕ^n with $n \geq 5$ excluded



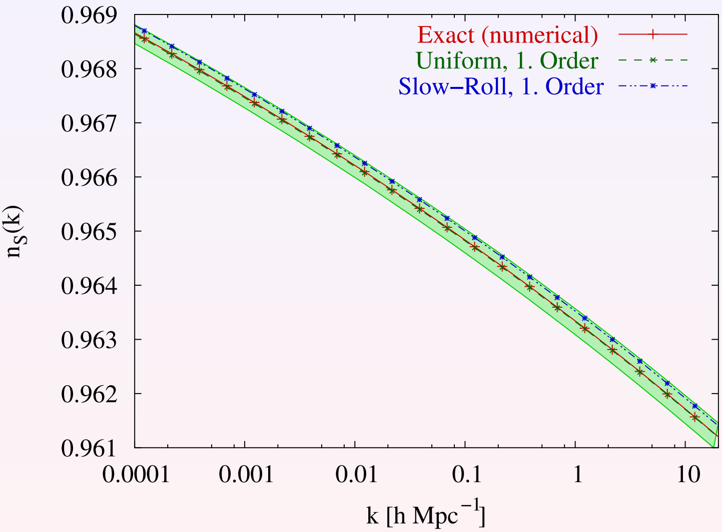
Features

- Slow-roll dominated dynamics $\rightarrow$ slow-roll expansion expected to work good for $m^2\phi^2$ potential (small n)
- $n_S < 1$, $n_T \neq n_S - 1$

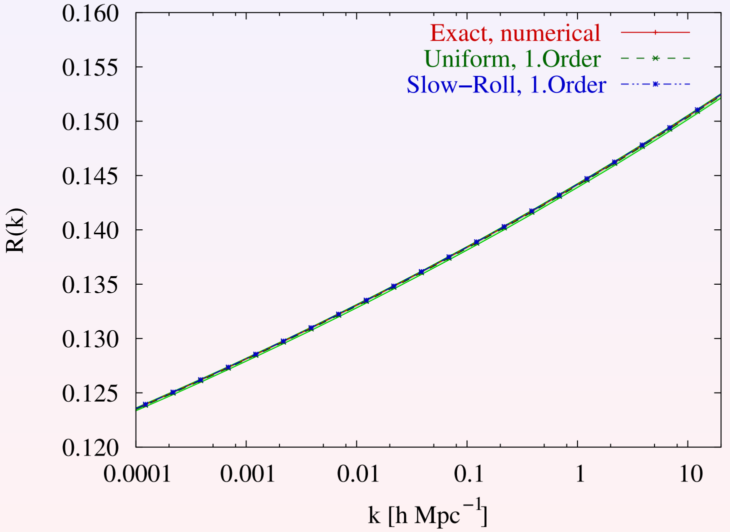
$m^2\phi^2$ chaotic inflation model – scalar power spectrum



$m^2\phi^2$ chaotic inflation model – scalar spectral index



$m^2\phi^2$ chaotic inflation model – ratio of tensor to scalar perturbations



Outline

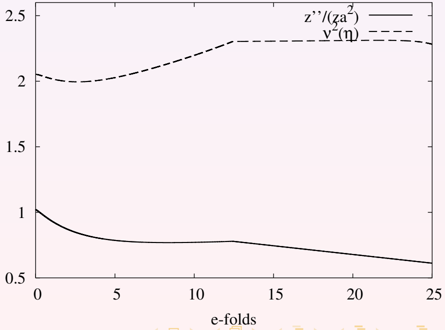
1. General Introduction and calculation of primordial calculations
2. The uniform approximation
   - General formalism
   - Leading order expressions with error bounds
   - Improvements and comments on errors
3. Results
   - Chaotic inflation
   - Phase transition model
4. Summary & Conclusions

Toy Model simulating a phase transition

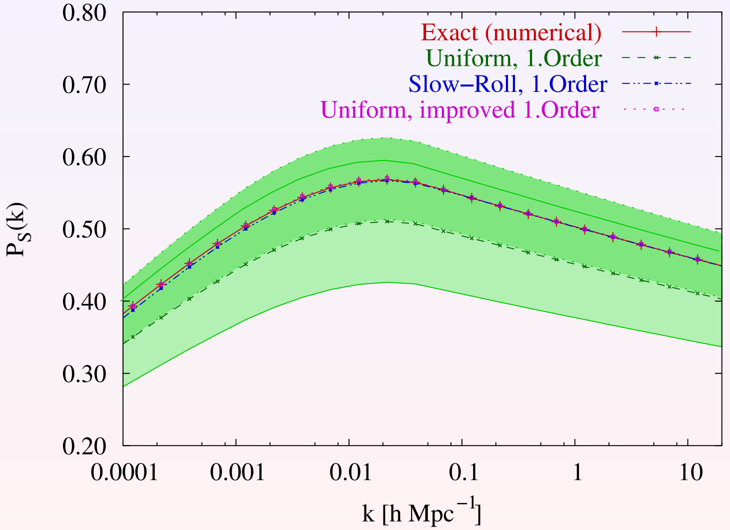
$$V_>(\phi) = \frac{1}{4}m^2\phi_*^2(\alpha-1) + \frac{2}{3}m^2\phi_*(1-\alpha)\phi + \frac{1}{2}\alpha m^2\phi^2 + \frac{1}{12\phi_*^2}m^2(1-\alpha)\phi^4 \quad (39)$$

$$V_<(\phi) = \frac{1}{2}m^2\phi^2 \quad (40)$$

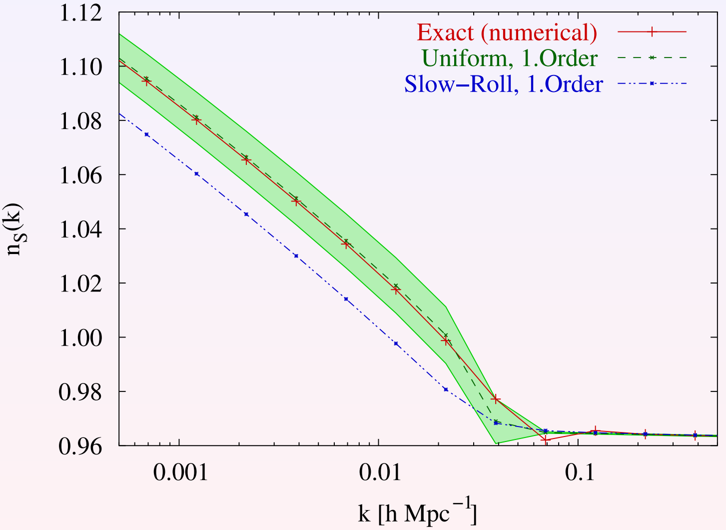
- $V_>(\phi_*) = V_<(\phi_*)$, $V'_>(\phi_*) = V'_<(\phi_*)$ $V''_>(\phi_*) = V''_<(\phi_*)$ $V'''_>(\phi_*) \neq V'''_<(\phi_*)$.
- $\phi > \phi_*$ "fast"-roll, $\phi < \phi_*$ slow-roll



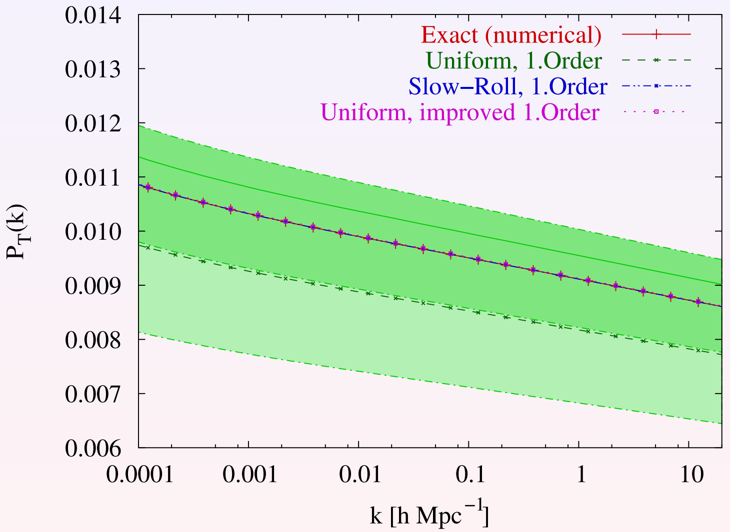
Scalar power spectrum



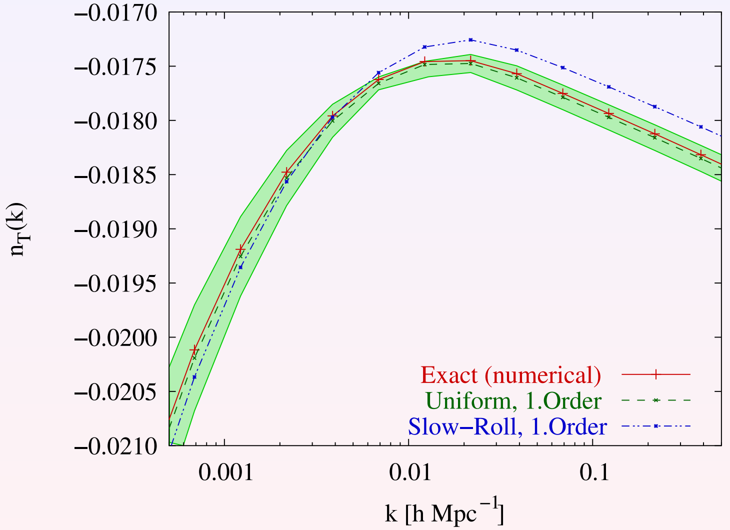
Scalar spectral index



Tensor power spectrum



Tensor spectral index



Summary & Conclusions

Summary & Conclusions

- Side-effect: We have developed an improved numerical mode-by-mode integration code (faster and more precise with the help of the uniform approximation)
- Leading order uniform approximation gives simple expressions in terms of integrals
- With a simple improvement of the leading order uniform approximation a general accuracy $\ll 1\%$ can be achieved, the spectral indices, the running and the tensor to scalar ratio being excellent already without the improvement.

Outlook

- Make the codes public available (with an interface to CMBFAST).
- Large parameter studies with plenty of inflation models

Thank you for your attention!

This presentation has been made with help of the LaTeX package **beamer**.
See `http://latex-beamer.sourceforge.net`