Entanglement and Minimal Hilbert Space in the Classical Dual States of Quantum Theory

Diego J. CIRILO-LOMBARDO

Special Astrophysical Observatory of Russian Academy of Sciences,

Radio Astrophysics Laboratory, AGN research group,

Nizhny Arkhyz, Russian Federation

M. V. Keldysh Institute of the Russian Academy of Sciences,

Federal Research Center-Institute of Applied Mathematics,

Miusskaya sq. 4, 125047 Moscow, Russian Federation and

CONICET-Universidad de Buenos Aires, Departamento de Fisica,

Instituto de Fisica Interdisciplinaria y Aplicada (INFINA), Buenos Aires, Argentina.

Norma G. SANCHEZ

The International School of Astrophysics Daniel Chalonge - Hector de Vega,

CNRS, INSU-Institut National des Sciences de l'Univers,

Sorbonne University. 75014 Paris, France.

Norma.Sanchez@obspm.fr

https://chalonge-devega.fr/sanchez

(Date august 2025)

Abstract

A precise physical description and understanding of **the classical dual content of quantum theory** is necessary in many disciplines today: from concepts and interpretation to quantum technologies and computation. In this paper we investigate Quantum Entanglement with the new approach APL Quantum 2, 016104 (2025) `https://doi.org/10.1063/5.0247698` on dual Classicalization. Thus, the results of this paper are twofold: **Entanglement and Classicalization and the relationship between them**. Classicalization truly occurs *only* under the action of the Metaplectic group $Mp(n)$ (Minimal Representation group, double covering of the Symplectic group). **The results of this paper are**: **(1)** We compute and analyze Entanglement for different types of coherent (coset and non coset) states and topologies: in the circle and the cylinder. We project the entangled wave functions onto the even (+) and odd (-) irreducible Hilbert $Mp(n)$ subspaces, and compute their square norms: **Entanglement Probabilities** P_{++}, P_{--}, P_{+-}, (eg in the same or in the different subspaces), and the Total sum of them. **(2) Entanglements in the circle**: (i) of orthogonal states are $\neq 0$ even for the control phase $\rho = 0$ and are preserved for any phase $\rho \neq \pi/2$. P_{+-} is separable (entanglement is broken) only if $\rho = 0$. (ii) For coincident states, entanglements are $\neq 0$ and separable for any control parameter $\rho \neq 0$. If $\rho = 0$, all three P_{++}, P_{--}, P_{+-} vanish. (iii) *The Entanglements of coset circle states* depend on the angles and the coherent complex displacement parameter α through a *single variable* z' which condition on the disk $|z'| < 1$ guarantees *both analyticity and normalization.* (iv) **Entanglement classicalization** in the circle is **stronger** for coset states than for non coset ones: a tail decreasing with increasing n is absent in the non coset case. **(3) Entanglement in the cylinder** depends weakly on the angles $(\varphi - \varphi')$ and **strongly classicalizes**: rapid exponential decay suppresion e^{-n^2} for large n, stronger than in the circle (be coset or not). It expresses in terms of **Theta functions** $\vartheta_2\left(0, e^{-8}\right)$ and $\vartheta_3\left(0, e^{-8}\right)$ in **all** classicalization projections in the degeneracy (equal states) limit. **(4)** Comparison with other (non $Mp(2)$) Entanglements, as the Schrodinger cat states, show very clear differences, (eg. Figs 4 and 5). **(5)** The **Antipodal and Non Antipodal Entanglements** (as regulated by the phase control parameter $\rho = \pi$ and $\rho = 0$ respectively), are clearly different in all types (circle, cylinder, coset or not) of **orthogonal** states studied here. *The Antipodal Entanglement is the Minimal.* These theoretical and conceptual results can be of experimental and practical real-world interest.

I. INTRODUCTION AND RESULTS

Quantum theory is at the center of science and technology today: The great success of Quantum theory and its deep and wide impact transcends physics and its applications: See for example Refs [1], [2], [3], [4] [5]. While the quantum wave nature of particles and the intrinsic quantum uncertainty principle were mainly about the first quantum revolution, the superposition principle and quantum entanglement are at the heart of the second quantum revolution at work. While quantum research is rapidly evolving in many different directions and disciplines, it is necessary to have a precise modern physical understanding and clear computational framework for **the classical dual content of quantum theory**: This impacts from the conceptual and quantum interpretation measurements until quantum technologies and computation research. Therefore, one fundamental question in Quantum theory with interest for its meaning, observation and experimental research is the following:

Under which conditions Quantum theory does **appear Classical, namely which are the Classical dual sectors of the Quantum world**.

Classical states are a particular case of Quantum theory and are dual states in the precise sense of the general classical-quantum duality of Nature. Recently, in ApL Quantum 2025 Ref [6], we provided a precise answer to this problem within a novel approach: The minimal group representation principle, which is uniquely realized by the Metaplectic group $Mp(2d)$. This group is the double covering of the Symplectic group $Sp(2d)$ and its action inmediately *classicalizes* the system. We performed an extensive study with different types of quantum states on the circle and on the cylinder from which emerged too that $Mp(2d)$ is the symmetry group of the general classical-quantum duality of Nature.

In this paper we go beyond in our study of the Classical sectors of Quantum theory by investigating the **Entanglement** for the quantum-classical dual states with topologies on the circle and on the cylinder and different types of coherent (coset and non coset) states. Thus, the results of this paper are on both:

Entanglement and Classicalization, and the relationship between them.

This is important too for the bridge between the classical and quantum information processings and relates to a real-world problem.

Discretization arises naturally and directly from the basic states of the metaplectic representations: the decomposition of the $Mp(2)$ group into its two irreducible representations span both: the *even* $\mid 2n \,\rangle$ and *odd* $\mid 2n \, + \, 1 \,\rangle$ states respectively, $(n \, = \, 1, \, 2, \, 3 \ldots)$ of the harmonic oscillator, totally covered by the metaplectic group. The two $Mp(2)$ irreducible subspaces contain both: the quantum and the classical dual sectors. For $n \to \infty$, the spectrum becomes naturally continuum as it must be.

A quantum state **completely classicalizes** its inherent quantum structure **only** under the action of the **Mp(n) (metaplectic) Group**, (Minimal Representation Group). Classicalization is explicit in the decreasing exponential factors for large n arising in the $Mp(2)$ projections of the states: screenings $e^{-2\,n}$, $e^{-(\,2n+1/2\,)}$ or $e^{-2\,n^2}$, $e^{-(\,2n+1/2\,)^2}$ (depending on the topology, circle or cylinder states respectively). The Classical-Quantum Duality is realized in the $Mp(n)$ symmetry because of the complete covering of the Hilbert space: Each of the two, *even* $\mathcal{H}_{(+)}$ and *odd* $\mathcal{H}_{(-)}$ sectors are local coverings, their sum being global, completely covering the *whole* Hilbert space $\mathcal{H} = \mathcal{H}_{(+)} \oplus \mathcal{H}_{(-)}$. The two $(+)$ and $(-)$ sectors are classical-quantum duals of each other and are **entangled**. This is also important in order to include gravity at the Planck scale and beyond: quantum space-time and classical-quantum gravity duality, Refs [12], [13], [14], [15], [16] which is general, irrespective of the number or type of space-time dimensions or manifolds (with or without compactifications).

In this paper our focus and results are the following:

(1) In analogy and for comparison with the entanglement of the standard coherent states in the number representation:

$$|n, m\rangle \; = \; \frac{1}{\sqrt{2}} \left(\, |n\rangle \otimes |m\rangle + |m\rangle \otimes |n\rangle \, \right),$$

we project the states (be coherent or not) into the number representation but through the basic states of the Metaplectic group $Mp(n)$, namely:

$$\left|\Psi^{(\pm)}\,(\omega)\,\Psi^{(\pm)}\,(\sigma)\right\rangle \; = \; \frac{1}{\sqrt{2}} \left(\left|\Psi^{(\pm)}\,(\omega)\right\rangle \otimes \left|\Psi^{(\pm)}\,(\sigma)\right\rangle + \left|\Psi^{(\pm)}\,(\sigma)\right\rangle \otimes \left|\Psi^{(\pm)}\,(\omega)\right\rangle \right) \qquad (1)$$

that is to say, with the application of the Minimal Group Representation which is precisely given by the Metaplectic group $Mp(n)$.

(2) The entanglement of the item (1) with the basic even (+) and odd (-) states of $Mp(n)$ is compared through the degree of purity with the case of the (even and odd) Schrodinger cat states:

$$|\alpha, -\alpha\rangle \; = \; \frac{1}{\sqrt{2}} \, (\, |\alpha\rangle \otimes |-\alpha\rangle \, + \, |-\alpha\rangle \otimes |\alpha\rangle \,) \, .$$

(3) The entanglement of two states with the topology of the circle (or any other state, coherent or not) when projected according to the Minimal Group Representation (Metaplectic group) provides two possibilities: This is because from the point of view of the metaplectic projection we have a general case entailing two analytical functions (σ, ω) $\rightarrow \Psi\,(\omega)\,\Psi\,(\sigma)$, that is:

$$\Psi\,(\omega) \begin{cases} \Psi^{(+)}\,(\omega) \rightarrow \; \mathcal{H}_+ \\ \\ \Psi^{(-)}\,(\omega) \longrightarrow \mathcal{H}_- \end{cases} \quad and \quad \Psi\,(\sigma) \begin{cases} \Psi^{(+)}\,(\sigma) \rightarrow \; \mathcal{H}_+ \\ \\ \Psi^{(-)}\,(\sigma) \longrightarrow \mathcal{H}_- \end{cases}$$

which gives precisely, (with the $Mp(n)$ schematic structure given above), states of the type of Eq.(1).

As a consequence of a characteristic property of the $Mp(n)$ group, this is the **only case** where **a single analytic function** allows to preserve the **uniqueness** of the mapping with a single analytic function, e.g. for $\sigma \longrightarrow \omega$, then $\Psi\,(\sigma) \rightarrow \Psi\,(\omega)$. Therefore:

$$\lim_{\sigma \rightarrow \omega} \left| \, \Psi^{(+)}\,(\omega)\,\Psi^{(-)}\,(\sigma) \, \right\rangle \sim \frac{1}{\sqrt{2}} \left[\, \left|\Psi^{(+)}\,(\omega\,)\right\rangle \otimes \left| \, \Psi^{(-)}\,(\omega\,)\right\rangle + \left|\Psi^{(+)}\,(\omega)\right\rangle \otimes \left|\Psi^{(-)}\,(\omega)\right\rangle \, \right]$$

This limit procedure must be taken explicitly in the projection for the cases of the states on the cylinder, as we will see here in Section III.

(4) We compute and analyze the different projections in each $Mp(2)$ even (+) and odd (-) sector of the entangled wave functions with topologies in the circle and in the cylinder. This provides for each topology three different entanglement possibilities depending on whether states are entangled in the same subspaces or in the crossed ones, namely (++), (- -), or (+ -), and the Total Entanglement of them. We compute too the corresponding square norm **Entanglement Probabilities** : even-even P_{++}, odd-odd P_{--}, and crossed P_{+-} Probabilities:

(5) For the circle (London) states : The Entanglements P_{++} or P_{--} of the same subspace states are similar (with *cosh* and *cos* functions for P_{++} and *sinh*, *sin* functions for P_{--}, while the crossed P_{+-} have both types of functions as it must be). We analyze the limits of coincident states $\Delta \to 0$, and of orthogonal states $\Delta \to \pi/2$, being $\Delta \equiv (\varphi - \varphi')$, and the role of the phase ρ (the admissible control parameter which is useful for the entanglement experimental realizations).

For the coincident states ($\Delta \to 0$)**:** The Entanglements P_{++}, P_{--}, P_{+-} are $\neq 0$ and separable (the entanglement is broken) for any phase $\rho \neq 0$. While, all three P_{++}, P_{--}, $P_{+-} = 0$ if $\rho = 0$. **For the orthogonal states** ($\Delta \to \pi/2$) **:** The Entanglements P_{++}, P_{--}, P_{+-} are $\neq 0$ even if $\rho = 0$. P_{++}, P_{--}, P_{+-} are preserved for any $\rho \neq \pi/2$. P_{+-} is separable (the entanglement is broken) only if $\rho = 0$. **Figures 2 and 3** and their captions illustrate some of these results.

(6) The Entanglements of the new coset circle states $|\,\varphi, \alpha\rangle$ do depend on the angles (φ, φ') and on the complex coherent parameter α through the **single variable** $z' = \omega\, e^{i\,(\varphi - \alpha^*/2)}$, which condition on the disk $|z'| < 1$ does remarkably guarantee both **analyticity and normalization**. This Entanglement clearly contains a decreasing tail as n increases showing that **the Entanglement classicalization** is **stronger** in the coset states than in the non coset (London) states, which we relate to the fact that these coset circle states are truly coherent analytic and normalizable states, (while the London, 't Hooft states are not).

(7) The Entanglement of the cylinder states: **(i)** Rapid decay through exponential suppresion factors $\propto e^{-n^2}$ for large n, showing that in all cases the Cylinder Entanglement **strongly classicalizes**. **(ii)** The low dependence of the Entanglement on the angular variables (contained in $\Delta = (\varphi - \varphi'))$ in the limiting case of degeneracy (equality) of the analytical functions describing the cylinder states : P_{++} and P_{--} are totally **independent of the angles** (φ, φ'), while the crossed P_{+-} entanglement does depend weakly on them. **(iii)** The cylinder Entanglements express in terms of **Theta functions** $\vartheta_2\,(0, e^{-8})$ and $\vartheta_3\,(0, e^{-8})$ in **all** classicalization projections in the degeneracy (equal states) limit. **(iv)** Cylinder topology does classicalize the Entanglement **stronger** than the circle topology. In **Figure 1** we *shortly summarize some of the main Entanglement and Classicalization features of this paper* both for the circle and cylinder topologies.

(8) Comparison of Entanglements are performed for completeness: For instance, $Mp(2)$ Entanglement projections are compared with other (non $Mp(2)$) projection states as the Schrodinger cat states: Differences are very clear as shown in **Figures 4 and 5** and their captions. The **Antipodal and and Non Antipodal Entanglements**, (as regulated by the phase control parameter $\rho = \pi$ and $\rho = 0$ respectively), are clearly different in the case of orthogonal states for **all types of orthogonal states** studied here (being Schrodinger cat or Mp(2) orthogonal states).

(9) Besides the theoretical and conceptual interest, the results of this paper should have **applications and implications** for experimental entanglement work, connections of classical and quantum treatments of information, interpretation of the quantum-classical interaction, the quantum interpretation measurements, classical or quantum optimization, to mention some of them. In Section VI Concluding Remarks we mention some **outlook** and implications of our results.

This paper is organized as follows: In Section II we describe and compute the quantum Entanglements of the phase states in the circle, its full global covering, the Entanglement Probabilities and their classicalization. In Section III we fully compute and analyze the Entanglements of the Cylinder States, their Probabilities and show how their classicalization occurs, finding very different and new properties with respect to the Entanglement and its classicalization in the phase space of the circle. In Section IV we perform the complete Entanglement for the general coset coherent states in the circle which allow to fully see the Entanglement and classicalizations conditions with respect to topology, analyticity in the disk of the wave functions and normalization, and compare with respect to the London (circle, phase space) states. In Section V we compare the Entanglements and Classicalization we found here with those of the Schrodinger cat states and discusses the implications of the $Mp(2)$ results of the previous sections with the cat state results. Sections VI summarizes Remarks and Conclusions.

	ENTANGLEMENT MAIN FEATURES AND CLASSICALIZATION
Circle (London) states U(1)	**(i)** The Entanglements P_{++} or P_{--} of the same subspace states (even(+) and odd(-)) are similar (with cosh and cos functions for P_{++} and sinh, sin functions for P_{--}, while the crossed P_{+-} have both types of functions as it must be). **(ii)** For the coincident states **($\Delta \to 0$):** The Entanglements P_{++}, P_{--}, P_{+-} are $\neq 0$ and separable (the entanglement is broken) for any phase $\rho \neq 0$. While, all three P_{++}, P_{--}, $P_{+-}=0$ if $\rho=0$. **(iii)** For the orthogonal states **($\Delta \to \pi/2$) :** The Entanglements P_{++}, P_{--}, P_{+-} are $\neq 0$ even if $\rho=0$. P_{++}, P_{--}, P_{+-} are preserved for any $\rho \neq \pi/2$. **(iv)** P_{+-} is separable (the entanglement is broken) only if $\rho=0$. **(v)** Antipodal Entanglement **$\rho= \pi$** is the Minimal
Cylinder states RxU(1)	**(i)** Rapid decay through exponential suppresion factors $\approx \exp(-n^2)$ for large n, showing that in all cases the Cylinder Entanglement strongly classicalizes. **(ii)** The low dependence of the Entanglement on the angular variables in the limiting case of degeneracy (equality) of the analytical functions describing the cylinder states : P_{++} and P_{--} are totally independent of the angles (φ,φ'), while the crossed P_{+-} entanglement does depend weakly on them. **(iii)** The cylinder Entanglements express in terms of Theta functions $\vartheta_2(0,e^{-8})$ and $\vartheta_3(0,e^{-8})$ in all classicalization projections in the degeneracy (equal states) limit. **(iv)** Cylinder topology does classicalize the Entanglement stronger than the circle topology. **(v)** Antipodal Entanglement **$\rho= \pi$** is the Minimal
Coset circle states E(2)/T2	**(i)** The Entanglements of the **new coset circle states $\|\varphi,\alpha$>** do depend on the angles (φ,φ') and on the complex coherent parameter α through the single variable $z'=\omega \exp\{i(\varphi-\alpha^*/2)\}$, which condition on the disk $\|z'\|<1$ does remarkably guarantee both analyticity and normalization. **(ii)** This Entanglement clearly contains a decreasing tail as n increases showing that the Entanglement classicalization is stronger in the coset states than in the non coset (London) states, which we relate to the fact that these coset circle states are truly coherent analytic and normalizable states, (while the London states are not). **(iii)** Antipodal Entanglement **$\rho= \pi$** is the Minimal

FIG. 1: **Some Main New Features of Entanglements and Classicalization of this Paper. Sections I and VI provide more Summary and explanation.**

II. ENTANGLEMENT OF COHERENT STATES IN THE CIRCLE

The classical states for the circle are coherent states in phase space, the angle variable,(or so called London states). Let us consider two of these states $|\varphi\rangle$, $|\varphi'\rangle$ which in terms of the harmonic oscillator eigenstates n, m are given by:

$$|\,\varphi\,\rangle \;=\; \frac{1}{2\pi}\sum_{n\,=\,0,\,1,\,2...} e^{-i\,\varphi\,n}\;|\,n\,\rangle \tag{2}$$

$$|\,\varphi'\,\rangle \;=\; \frac{1}{2\pi}\sum_{m\,=\,0,\,1,\,2...} e^{-i\,\varphi'\,m}\;|\,m\,\rangle \tag{3}$$

An entangled state can be defined naturally as

$$|\,\Phi\,\rangle \;\equiv\; \frac{1}{\sqrt{2}}\,\big(\,|\varphi\rangle\otimes|\varphi'\rangle \,+\, e^{i\rho}\;|\varphi'\rangle\otimes|\varphi\rangle\,\big) \tag{4}$$

where the order of the states in Eq. (4) indicates the exchange of the angular coordinate (in this case φ, φ'), and the phase ρ is an admissible control parameter which is very useful for the experimental realization of entanglement.

Now, following the Minimal Group Representation consisting of the metaplectic group $Mp(2)$, we make the metaplectic projection of the entangled state. To this end, we perform the $Mp(2)$ product state :

$$\big|\,\Psi^{(\pm)}\,(\omega)\,\Psi^{(\pm)}\,(\sigma)\,\big\rangle \,\in\, Mp\,(2)\,,$$

$$\big|\,\Psi^{(\pm)}\,(\omega)\,\Psi^{(\pm)}\,(\sigma)\,\big\rangle \;=\; \frac{1}{\sqrt{2}}\,\big[\,\big|\Psi^{(\pm)}\,(\omega)\big\rangle\otimes\big|\Psi^{(\pm)}\,(\sigma)\big\rangle + \big|\Psi^{(\pm)}\,(\sigma)\big\rangle\otimes\big|\Psi^{(\pm)}\,(\omega)\big\rangle\,\big]$$

The even $(+)$ and odd $(-)$ $Mp(2)$ states (irreducible Metaplectic representations) are given by:

$$\langle\,\varphi|\,\big|\Psi^{(\pm)}\,(\omega)\,\big\rangle = \begin{cases} \left(1-|\omega|^2\right)^{1/4} \displaystyle\sum_{n\,=\,0,1,2..} \frac{\left(\omega e^{i\varphi}/2\right)^{2n}}{\sqrt{2n!}} & (+)\text{: even states} \\[2ex] \left(1-|\omega|^2\right)^{3/4} \displaystyle\sum_{n\,=\,0,1,2..} \frac{\left(\omega e^{i\varphi}/2\right)^{2n+1}}{\sqrt{(2n+1)!}} & (\text{-})\text{: odd states} \end{cases} \tag{5a}$$

Therefore, the total or complete projected state $\langle\, \varphi |\, |\, \Psi\,(\omega)\,\rangle$ is given by:

$$\langle\, \varphi |\, |\, \Psi\,(\omega)\,\rangle \;=\; \langle\, \varphi |\, \big|\, \Psi^{(+)}\,(\omega)\,\big\rangle \;+\; \langle\, \varphi |\, \big|\, \Psi^{(-)}\,(\omega)\,\big\rangle \tag{6}$$

$$\langle\, \varphi\, |\, |\, \Psi\,(\omega)\,\rangle \;=\; \frac{\left(1-|z|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z/2)^{2n}}{\sqrt{(2n)\,!}} \left[1 \;+\; \left(1-|z|^2\right)^{1/2} \frac{(z/2)}{\sqrt{2n+1}} \right] \tag{7}$$

where $z \;=\; \omega\, e^{i\varphi}$.

Now, having the above expressions into account, we define in the Bargmann representation the following functions in order to perform the respective projections:

$$z_1 \equiv \omega e^{i\varphi} \to \langle\, \varphi |\, \big|\Psi^{(\pm)}\,(\omega)\,\big\rangle \equiv \Psi^{(\pm)}\,(z_1)\,, \qquad z_1' \equiv \omega e^{i\varphi'} \to \langle\, \varphi' |\, \big|\Psi^{(\pm)}\,(\omega)\,\big\rangle \equiv \Psi^{(\pm)}\,(z_1')$$

$$z_2 \equiv \sigma e^{i\varphi} \to \langle\, \varphi |\, \big|\Psi^{(\pm)}\,(\sigma)\,\big\rangle \equiv \Psi^{(\pm)}\,(z_2)\,, \qquad z_2' \equiv \sigma e^{i\varphi'} \to \langle\, \varphi' |\, \big|\Psi^{(\pm)}\,(\sigma)\,\big\rangle \equiv \Psi^{(\pm)}\,(z_2')$$

Therefore :

$$\left\langle \Psi^{(\pm)}\,(\omega)\,\Psi^{(\pm)}\,(\sigma) \right| |\Phi\rangle \;=\; \frac{1}{2} \left[\Psi^{(\pm)}\,(z_1) \otimes \Psi^{(\pm)}\,(z_2') \;+\; e^{i\rho}\Psi^{(\pm)}\,(z_1') \otimes \Psi^{(\pm)}\,(z_2) \right]$$

Or, due to Eq. (7) using the complete projected states, we have:

$$\begin{aligned}
\langle \Psi\,(\omega)\,\Psi\,(\sigma)|\, |\Phi\rangle = \frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{1/4} \sum_{n,m=0,1,2...} \frac{(\omega^*/2)^{2n}}{\sqrt{2n!}} \frac{(\sigma^*/2)^{2m}}{\sqrt{2m!}} \times \\
\times \left[e^{-2i(\varphi n+\varphi' m)} \left(1+\left(1-|\omega|^2\right)^{1/2} \frac{\left(\omega^* e^{-i\varphi}/2\right)}{\sqrt{2n+1}}\right) \left(1+\left(1-|\sigma|^2\right)^{1/2} \frac{\left(\sigma^* e^{-i\varphi'}/2\right)}{\sqrt{2m+1}}\right) + \right. \\
\left. e^{i\rho} e^{-2i(\varphi' n+\varphi m)} \left(1+\left(1-|\omega|^2\right)^{1/2} \frac{\left(\omega^* e^{-i\varphi'}/2\right)}{\sqrt{2n+1}}\right) \left(1+\left(1-|\sigma|^2\right)^{1/2} \frac{\left(\sigma^* e^{-i\varphi}/2\right)}{\sqrt{2m+1}}\right) \right]
\end{aligned} \tag{8}$$

A. Entanglement and Minimal Group Representation

The minimal group representation which is provided by the metaplectic group implies for the circle states the crossed even (+) odd (-) explicit projection :

$$\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\omega)\right|\left|\Phi\right\rangle = \frac{1}{2}\left(1-|\omega|^2\right)$$

$$\sum_{n,m=0,1,2...}\left[\frac{\left(\omega^*/2\right)^{2(n+m)+1}}{\sqrt{(2n)!\,(2m+1)!}}\left(e^{-i2n\varphi}e^{-i(2m+1)\varphi'}+e^{i\rho}e^{-i2n\varphi'}e^{-i(2m+1)\varphi}\right)\right]$$

and its square norm Probability P_{+-} :

$$P_{+-}=\left\|\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\omega)\right|\left|\Phi\right\rangle\right\|^2 = \frac{1}{2}\left(1-|\omega|^2\right)^2\left\{\cosh\left(\frac{|\omega|^2}{4}\right)\sinh\left(\frac{|\omega|^2}{4}\right)+\right.$$

$$\left.+\cosh\left(\frac{|\omega|^2}{4}\cos\Delta\right)\sinh\left(\frac{|\omega|^2}{4}\cos\Delta\right)\cos\rho+\cos\left(\frac{|\omega|^2}{4}\sin\Delta\right)\sin\left(\frac{|\sigma|^2}{4}\sin\Delta\right)\sin\rho\right\}$$

In what follows, we analyze the particular projections in each sector (even and odd) of the entangled wave functions together with the corresponding square norm Probabilities

B. Entanglement Probabilities of the Even (+) and Odd (-) Hilbert space sectors

Considering that the basic states of $Mp(n)$ solve the identity in each irreducible subspace representation $s = 1/4,\, 3/4,$ of the total Hilbert space, we will show the corresponding expressions of the **Entanglement Probabilities** from the general wave function Eq. (8) projected in each subspace for simplicity

- Entanglement Probability P_{++} of the Even (++) Sectors:

The explicit projected (+)(+) entangled expression of the London states wave function is:

$$\left\langle \Psi^{(+)}(\omega)\,\Psi^{(+)}(\sigma)\right|\left|\Phi\right\rangle = \frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{1/4}$$

$$\sum_{n,m=0,1,2...}\left[\frac{\left(\omega^*e^{-i\varphi}/2\right)^{2n}}{\sqrt{2n!}}\frac{\left(\sigma^*e^{-i\varphi'}/2\right)^{2m}}{\sqrt{2m!}}+\frac{\left(\omega^*e^{-i\varphi'}/2\right)^{2n}}{\sqrt{2n!}}\frac{\left(\sigma^*e^{-i\varphi}/2\right)^{2m}}{\sqrt{2m!}}e^{i\rho}\right]$$

Defining $\Delta\equiv(\varphi-\varphi')$ in the limit for the same variable : $\varphi\rightarrow\varphi'$, *ie* $\Delta\rightarrow 0$, and for the orthogonal states, ie $\Delta\rightarrow\ \pi/2$, the entanglement Probabilities P_{++} (strictly the norm square) considerably simplify as given explicitly in the Appendix I whose limits of interest $\Delta \to 0$ (complete degeneration) and $\Delta \to \pi/2$ are the following:

$$\lim_{\Delta\to 0} P_{++} \;=\; \frac{1}{2}\sqrt{\left(1-|\omega|^2\right)\left(1-|\sigma|^2\right)}\cosh\left(\frac{|\omega|^2}{4}\right)\cosh\left(\frac{|\sigma|^2}{4}\right)(1-\cos\rho)$$

$$\lim_{\Delta\to\pi/2} P_{++} \;=\; \frac{1}{2}\sqrt{\left(1-|\omega|^2\right)\left(1-|\sigma|^2\right)}\left\{\cosh\left(\frac{|\omega|^2}{4}\right)\cosh\left(\frac{|\sigma|^2}{4}\right)+\cos\left(\frac{|\omega|^2}{4}\right)\cos\left(\frac{|\sigma|^2}{4}\right)\cos\rho\right\}$$

We see the role played by the phase ρ in the entanglement P_{++} which makes it non zero iff $\rho \neq 0$ for the case of the coincident states ($\Delta \to 0$), while the entanglement is non-zero even if $\rho = 0$ for the orthogonal states ($\Delta \to \pi/2$).

For $\rho = 0$, we see that:

$$\lim_{\Delta\to 0} P_{++} \;=\; 0,$$

and

$$\lim_{\Delta\to \pi/2} P_{++} \;=\; \frac{1}{2}\sqrt{\left(1-|\omega|^2\right)\left(1-|\sigma|^2\right)}\cosh\left(\frac{|\omega|^2}{4}\right)\cosh\left(\frac{|\sigma|^2}{4}\right)$$

Therefore, the *state is separable*, that is to say, the *entanglement is broken* in this case.

- Entanglement Probability P_{+-} of the Even-Odd (+-) Sectors:

The London states crossed projected into the even (+) and odd (-) Metaplectic states have the following explicit expression:

$$\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle \;=\; \frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{3/4}$$

$$\sum_{n,\,m\,=\,0,1,2\ldots}\left[\frac{\left(\omega^* e^{-i\varphi}/2\right)^{2n}}{\sqrt{(2n)!}}\,\frac{\left(\sigma^* e^{-i\varphi'}/2\right)^{2m+1}}{\sqrt{(2m+1)!}} \;+\; e^{i\rho}\,\frac{\left(\omega^* e^{-i\varphi'}/2\right)^{2n}}{\sqrt{2n!}}\,\frac{\left(\sigma^* e^{-i\varphi}/2\right)^{2m+1}}{\sqrt{(2m+1)!}}\right]$$

Now we see the role played by the phase ρ in the entanglement probability P_{+-} in the corresponding limits:

In the coincident limit :

$$\lim_{\Delta\to 0} P_{+-} \;=\; \frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{3/2}\cosh\left(\frac{|\omega|^2}{4}\right)\sinh\left(\frac{|\sigma|^2}{4}\right)(1-\cos\rho)$$

We see the above expression is separable (the entanglement is broken) for any value of the phase $\rho \neq 0$ while $\lim_{\Delta \to 0} P_{+-} = 0$ for $\rho = 0$.

In the orthogonal limit :

$$\lim_{\Delta \to \pi/2} P_{+-} = \frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{3/2}\left\{\cosh\left(\frac{|\omega|^2}{4}\right)\sinh\left(\frac{|\sigma|^2}{4}\right)+\cos\left(\frac{|\omega|^2}{4}\right)\sin\left(\frac{|\sigma|^2}{4}\right)\sin\rho\right\}$$

In this case the entanglement is preserved for any value of $\rho \neq 0$. Only it is separable (the entanglement is broken) for $\rho = 0$.

- Entanglement Probability P_{--} of the Odd (- -) Sectors :

For the odd Hilbert space sector, the projection of the circle states is given by:

$$\left\langle \Psi^{(-)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle = \frac{1}{2}\left(1-|\omega|^2\right)^{3/4}\left(1-|\sigma|^2\right)^{3/4}$$

$$\sum_{n,\,m\,=\,0,\,1,\,2\,...}\left[\frac{\left(\omega^* e^{i\varphi}/2\right)^{2n+1}}{\sqrt{(2n+1)!}}\,\frac{\left(\sigma^* e^{i\varphi'}/2\right)^{2m+1}}{\sqrt{(2m+1)!}}+\frac{\left(\omega^* e^{i\varphi'}/2\right)^{2n+1}}{\sqrt{(2n+1)!}}\,\frac{\left(\sigma^* e^{i\varphi}/2\right)^{2m+1}}{\sqrt{(2m+1)!}}e^{i\rho}\right]$$

Now, we see the role played by the phase ρ in the entanglement probability P_{--} from the corresponding limits:

In the coincident limit:

$$\lim_{\Delta \to 0} P_{--} = \frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{3/2}\sinh\left(\frac{|\omega|^2}{4}\right)\sinh\left(\frac{|\sigma|^2}{4}\right)(1-\cos\rho)$$

In the limit of coincident states P_{--} is separable (the entanglement is broken) for any value of the phase $\rho \neq 0$, while it is vanishes for $\rho = 0$:

$$\lim_{\Delta \to 0} P_{--} = 0 \quad \text{when} \quad \rho = 0$$

In the orthogonal limit:

$$\lim_{\Delta \to \pi/2} P_{--} = \frac{1}{2}\left[\left(1-|\omega|^2\right)\left(1-|\sigma|^2\right)\right]^{3/2}\left\{\sinh\left(\frac{|\omega|^2}{4}\right)\sinh\left(\frac{|\sigma|^2}{4}\right)+\sin\left(\frac{|\omega|^2}{4}\right)\sin\left(\frac{|\sigma|^2}{4}\right)\cos\rho\right\}$$

In this case the entanglement is preserved for any value of $\rho \neq \pi/2$ (within the respective modulus), while the entanglement is broken for $\rho = \pi/2$.

It must be noticed, and as it can be seen in Figure 1: For the coincident states $\Delta = 0$, the **antipodal entanglement** (regulated by the control parameter $\rho = \pi$) generates a effect of **minimum entanglement** probability in **all** the Mp(2) projections . This is in marked contrast to the entanglement of orthogonal states $\Delta = \pi/2$ in Figure 3 where the antipodal entanglement ($\rho = \pi$) is relevant, inverting the maxima of the probability.

C. Full Entangled Probability

For completeness, to calculate the square norm in the full entangled state, we introduce a polar decomposition for the functions of the states of Mp(2), namely

$$\omega = |\omega| \, e^{i\theta_1}, \qquad \sigma = |\sigma| \, e^{i\theta_2}$$

Then:

$$\begin{aligned}
\| \langle \Psi(\omega) \, \Psi(\sigma) | \, |\Phi\rangle \|^2 = \frac{1}{4} \left(1 - |\omega|^2\right)^{1/2} \left(1 - |\sigma|^2\right)^{1/2} \sum_{n,\, m = 0,1,2\ldots} \frac{\left(|\omega|^2 /4\right)^{2n}}{2n!} \frac{\left(|\sigma|^2 /4\right)^{2m}}{2m!} \times \\
\times \left(1 + \left(1 - |\omega|^2\right)^{1/2} \frac{\cos(\theta_1 + \varphi)}{\sqrt{2n+1}} |\omega| + \frac{\left(1 - |\omega|^2\right)}{4(2n+1)} |\omega|^2 \right) \\
\left(1 + \left(1 - |\sigma|^2\right)^{1/2} \frac{\cos(\theta_2 + \varphi')}{\sqrt{2m+1}} |\sigma| + \frac{\left(1 - |\sigma|^2\right)}{4(2m+1)} |\sigma|^2 \right) + \\
+ \left(1 + \left(1 - |\omega|^2\right)^{1/2} \frac{\cos(\theta_1 + \varphi')}{\sqrt{2n+1}} |\omega| + \frac{\left(1 - |\omega|^2\right)}{4(2n+1)} |\omega|^2 \right) \\
\left(1 + \left(1 - |\sigma|^2\right)^{1/2} \frac{\cos(\theta_2 + \varphi)}{\sqrt{2m+1}} |\sigma| + \frac{\left(1 - |\sigma|^2\right)}{4(2m+1)} |\sigma|^2 \right) + \\
+ \cos\left(\rho + (\varphi' - \varphi)(n - m)\right) \left(AB - CD \right)
\end{aligned}$$

where :

$$A = \frac{1}{2}\Bigg[2 + \left(1-|\omega|^2\right)^{1/2} \frac{cos\,(\theta_1+\varphi)}{\sqrt{2n+1}}\,|\omega| + \left(1-|\sigma|^2\right)^{1/2} \frac{\cos\,(\theta_2+\varphi)}{\sqrt{2m+1}}\,|\sigma| + \frac{\left(1-|\sigma|^2\right)^{1/2}\left(1-|\omega|^2\right)^{1/2}}{\sqrt{2n+1}\,\sqrt{2m+1}} \frac{|\omega|\,|\sigma|}{2}\cos\,(\theta_1-\theta_2)\Bigg]$$

$$B = \frac{1}{2}\Bigg[2 - \left(1-|\omega|^2\right)^{1/2} \frac{\cos\,(\theta_1+\varphi')}{\sqrt{2n+1}}\,|\omega| + \left(1-|\sigma|^2\right)^{1/2} \frac{\cos\,(\theta_2+\varphi')}{\sqrt{2m+1}}\,|\sigma| + \frac{\left(1-|\sigma|^2\right)^{1/2}\left(1-|\omega|^2\right)^{1/2}}{\sqrt{2n+1}\sqrt{2m+1}} \frac{|\omega|\,|\sigma|}{2}\cos\,(\theta_2-\theta_1)\Bigg]$$

$$C = \frac{1}{2}\Bigg[\left(1-|\omega|^2\right)^{1/2} \frac{\sin\,(\theta_1+\varphi)}{\sqrt{2n+1}}\,|\omega| - \left(1-|\sigma|^2\right)^{1/2} \frac{\sin\,(\theta_2+\varphi)}{\sqrt{2m+1}}\,|\sigma| + \frac{\left(1-|\sigma|^2\right)^{1/2}\left(1-|\omega|^2\right)^{1/2}}{\sqrt{2n+1}\sqrt{2m+1}} \frac{|\omega|\,|\sigma|}{2}\sin\,(\theta_1-\theta_2)\Bigg]$$

$$D = \frac{1}{2}\Bigg[- \left(1-|\omega|^2\right)^{1/2} \frac{\sin\,(\theta_1+\varphi')}{\sqrt{2n+1}}\,|\omega| + \left(1-|\sigma|^2\right)^{1/2} \frac{\sin\,(\theta_2+\varphi')}{\sqrt{2m+1}}\,|\sigma| + \frac{\left(1-|\sigma|^2\right)^{1/2}\left(1-|\omega|^2\right)^{1/2}}{\sqrt{2n+1}\sqrt{2m+1}} \frac{|\omega|\,|\sigma|}{2}\sin\,(\theta_2-\theta_1)\Bigg]$$

As we can see, in the general case the total output of the London entanglement states is evidently manifested with the highest degree of entanglement.

III. ENTANGLEMENT OF COHERENT STATES IN THE CYLINDER

As we pointed out recently in Ref [6] in order to introduce the Coherent States for a quantum particle on the cylinder geometry, it is possible to follow the Barut–Girardello construction and seek the Coherent State as the solution of the eigenvalue equation:

$$X\,|\,\xi\,\rangle \;=\; \xi|\,|\,\xi\,\rangle$$

with the complex parameter ξ, similarly to the standard case, and

$$X \;=\; e^{\,i\,\left(\widehat{\varphi}+\widehat{J}\right)}$$

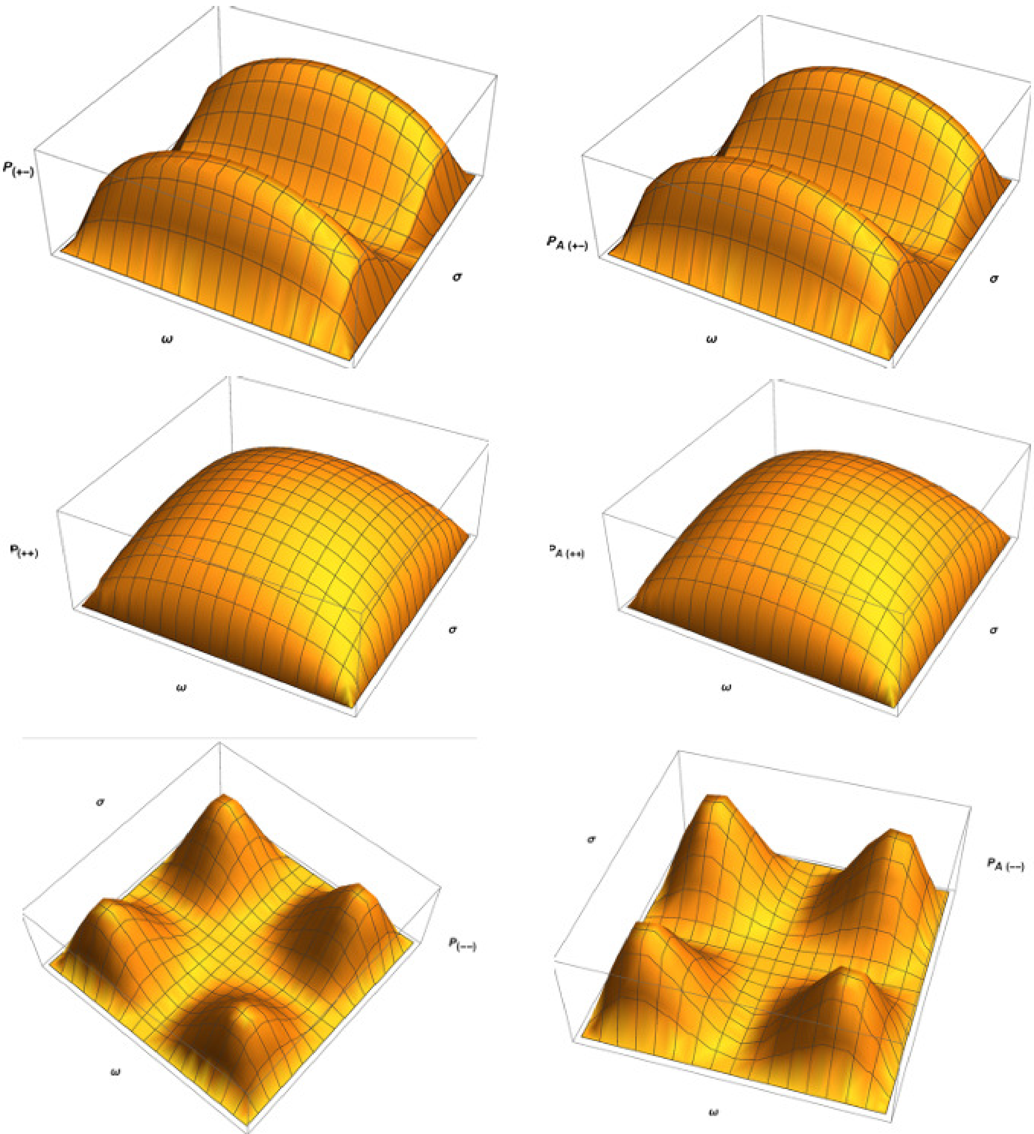

FIG. 2: **Probabilities of the Entanglement** of two circle (London) states with $\Delta = 0$ ($\varphi = \varphi'$): **coincident states** projected onto the Metaplectic group (Minimal Representation Group): **Classicalization**: Left side with the control entanglement phase $\rho = 0$. Right side with $\rho = \pi$: e.g **Antipodal Entanglement** . In this case, **the Antipodal** condition (regulated by the control parameter $\rho = \pi$) does not modify essentially any of the Entanglement Mp (2) Classicalizations.

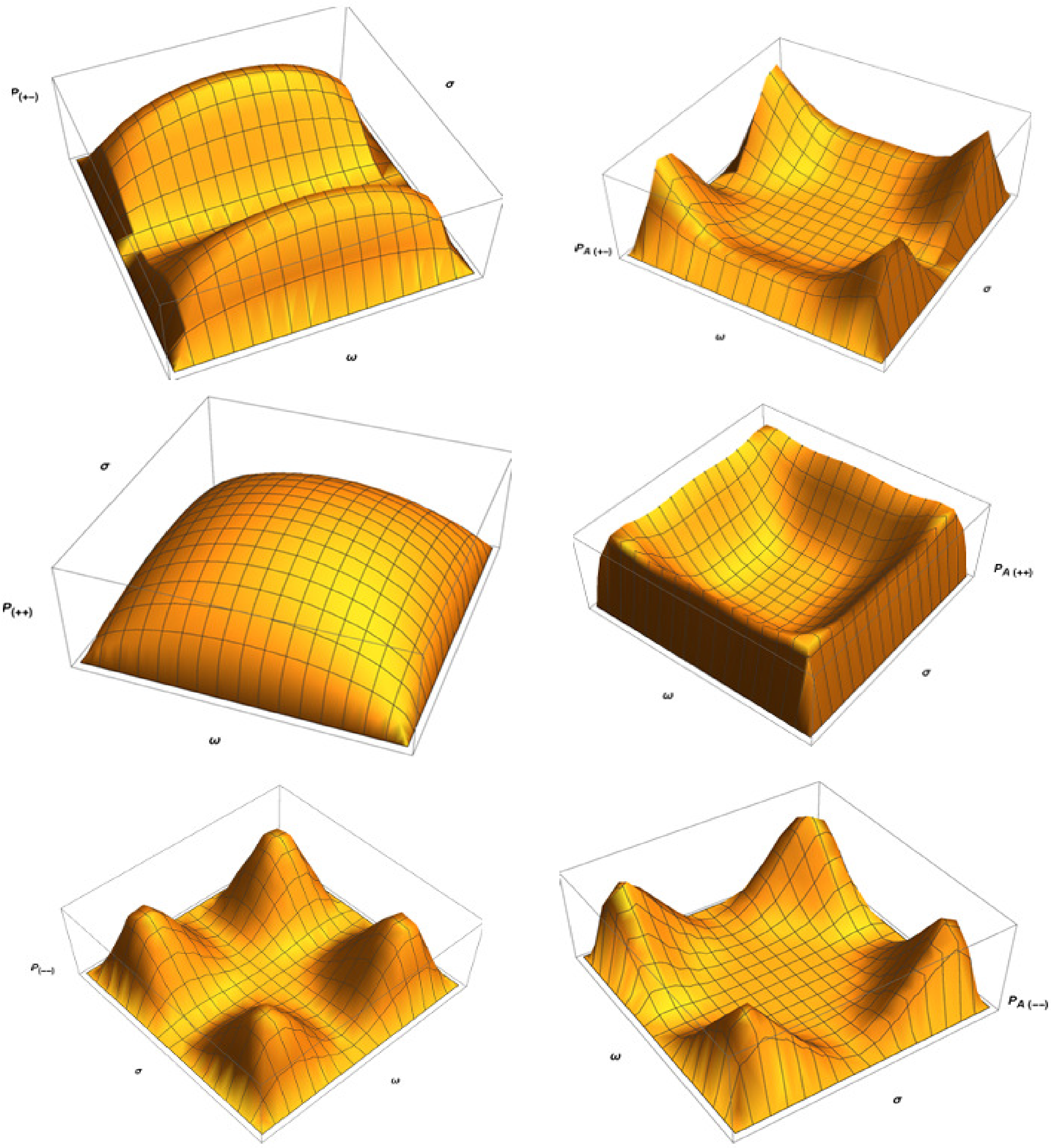

FIG. 3: **Probabilities of the Entanglement** of two circle London states $\Delta = \pi/2$ ($\varphi = \varphi' + \pi/2$): **orthogonal states** projected onto the Metaplectic group (Minimal Representation Group): **Classicalization**. Left side entanglement with the control paramenter phase $\rho = 0$. Right side with $\rho = \pi$: **Antipodal Entanglament**. Here the antipodal condition is relevant, inverting the maxima of the Entanglement Probabilities.

In order to analyze the Coherent State of a particle in the cylinder in the context of the Minimal Group Representation, we express the coherent states as:

$$| \xi \rangle \;=\; \sum_{j\,=\,-\infty}^{\infty} e^{(\,l-i\,\varphi\,)j}\, e^{-j^2/2}\; |\, j\, \rangle \tag{9}$$

(i) If $|\, j\,\rangle \sim |\, n\,\rangle$, and for the Metaplectic $s \;=\; 1/4$ states $\left|\, \Psi^{(+)}\,(\omega)\, \right\rangle$:

$$\langle\, \xi\, |\, \left|\, \Psi^{(+)}\,(\omega)\, \right\rangle = \left(1 - |\omega|^2\right)^{1/4} \sum_{m\,=\,-\infty}^{\infty}\; \sum_{n\,=\,0,1,2...} \frac{(\omega/2)^{2n}}{\sqrt{2n\,!}}\; e^{(l-i\varphi)\,m}\; e^{-2m^2}\; \langle m|\, |2n\rangle$$

$$=\; \left(1 - |\omega|^2\right)^{1/4} \sum_{n\,=\,0,1,2..} \frac{\left(\omega\, e^{\,(l-i\varphi)}/2\right)^{2n}}{\sqrt{2n\,!}}\; e^{-2n^2}$$

(ii) Similarly, for the $s \;=\; 3/4$ Metaplectic states $\left|\, \Psi^{(-)}\,(\omega)\, \right\rangle$:

$$\langle\, \xi\, |\, \left|\, \Psi^{(-)}\,(\omega)\, \right\rangle = \left(1 - |\omega|^2\right)^{3/4} \sum_{m\,=-\infty}^{\infty}\; \sum_{n\,=\,0,1,2...} \frac{(\omega/2)^{2n+1}}{\sqrt{(2n+1)\,!}}\; e^{(l-i\varphi)\,m}\; e^{-m^2/2}\; \langle m|\, |2n+1\rangle$$

$$= \left(1 - |\omega|^2\right)^{3/4} \sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{(l-i\varphi)}/2\,\right)^{2n+1}}{\sqrt{(2n+1)\,!}}\, e^{-(2n+1)^2/2}$$

We see that the scalar product projections taken with the cilinder $\langle\, \xi\, |$ space configuration states are similar to the projections taken with the circle $\langle\, \varphi\, |$ phase space states, but *in contrast* they contain weight functions: e^{-2n^2} and $e^{-(2n+1)^2/2}$, which *drastically attenuate* the scalar products when $n \to \infty$:

$$\langle\, \xi\, |\, \left|\, \Psi^{(\pm)}\,(\omega)\, \right\rangle = \begin{cases} \left(1 - |\omega|^2\right)^{1/4} \displaystyle\sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{(l-i\varphi)}/2\,\right)^{2n}}{\sqrt{2n\,!}}\; e^{\,-2n^2} & \text{even states} \\ \left(1 - |\omega|^2\right)^{3/4} \displaystyle\sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{\,(l-i\varphi)}/2\,\right)^{2n+1}}{\sqrt{(2n+1)\,!}}\; e^{-\,(2n+1)^2/2} & \text{odd states} \end{cases} \tag{10a}$$

Consequently, for the **total state** :

$$|\, \Psi\,(\omega)\, \rangle \;=\; \left|\, \Psi^{(+)}\,(\omega)\, \right\rangle \;+\; \left|\, \Psi^{\,(-)}\,(\omega)\, \right\rangle, \tag{11}$$

we have:

$$\langle\,\xi\,|\,|\,\Psi\,(\omega)\,\rangle = \left(1-|\omega|^2\right)^{1/4} \sum_{n\,=\,0,1,2...} \frac{\left(\omega\, e^{(l-i\varphi)}/2\right)^{2n}}{\sqrt{2n\,!}}\, e^{-2\,n^2} \left[1+\left(1-|\omega|^2\right)^{1/2} \frac{\omega\, e^{\,(l-i\varphi)}}{\sqrt{2n+1}}\, e^{-(2n+1/2)}\right] \tag{12}$$

In order to simplify, in analogy with the previous case, we define in the Bargmann representation

$$z_1 \equiv \omega e^{(l+i\varphi)} \to \langle\,\xi|\,\left|\Psi^{(\pm)}\,(\omega)\,\right\rangle \equiv \Psi^{(\pm)}\,(z_1)\,, \qquad z_1' \equiv \omega e^{(l'+i\varphi')} \to \langle\,\xi'|\,\left|\Psi^{(\pm)}\,(\omega)\,\right\rangle \equiv \Psi^{(\pm)}\,(z_1')$$

$$z_2 \equiv \sigma e^{(l+i\varphi)}, \to \langle\,\xi|\,\left|\Psi^{(\pm)}\,(\sigma)\,\right\rangle \equiv \Psi^{(\pm)}\,(z_2)\,, \qquad z_2' \equiv \sigma e^{(l'+i\varphi')} \to \langle\,\xi'|\,\left|\Psi^{(\pm)}\,(\sigma)\,\right\rangle \equiv \Psi^{(\pm)}\,(z_2')$$

Then, the wave entangled projected state is expressed as :

$$\left\langle\Psi^{(\pm)}\,(\omega)\,\Psi^{(\pm)}\,(\sigma)\right|\,|\Phi\rangle \;=\; \frac{1}{2}\,\left(\,\Psi^{(\pm)}\,(z_1)\otimes\Psi^{(\pm)}\,(z_2')\;+\;e^{i\rho}\,\Psi^{(\pm)}\,(z_1')\otimes\Psi^{(\pm)}\,(z_2)\,\right)$$

Consequently, the full projected entangled wave function takes the form:

$$\langle\Psi\,(\omega)\,\Psi\,(\sigma)|\,|\Phi\rangle = \frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{1/4} \sum_{n,m=1,2...} \frac{\left(\omega^*/2\right)^{2n}}{\sqrt{2n!}}\frac{\left(\sigma^*/2\right)^{2m}}{\sqrt{2m!}} e^{-2\left(n^2+m^2\right)}\times \tag{13}$$

$$\times\left[e^{2((l-i\varphi)n+(l'-i\varphi')m)}\left(1+\left(1-|\omega|^2\right)^{1/2}\frac{\left(\omega^*\,e^{\,(l-i\varphi)}/2\right)}{\sqrt{2n+1}}\,e^{-(2n+1/2)}\right)\right.$$
$$\left(1+\left(1-|\sigma|^2\right)^{1/2}\frac{\left(\sigma^*\,e^{\,(l'-i\varphi')}/2\right)}{\sqrt{2m+1}}\,e^{-(2m+1/2)}\right) \tag{14}$$

$$+e^{i\rho}e^{2((l-i\varphi)m+(l'-i\varphi')n)}\left(1+\left(1-|\omega|^2\right)^{1/2}\frac{\left(\omega^*\,e^{\,(l'-i\varphi')}/2\right)}{\sqrt{2n+1}}\,e^{-(2n+1/2)}\right)$$
$$\left.\left(1+\left(1-|\sigma|^2\right)^{1/2}\frac{\left(\sigma^*\,e^{\,(l-i\varphi)}/2\right)}{\sqrt{2m+1}}\,e^{-(2m+1/2)}\right)\right] \tag{15}$$

A. Entanglement and Minimal Group Representation in the Cylinder

In this case, we will analyze the unique and fundamental projection due to the underlying metaplectic structure for each of the particular sectors of the full Hilbert space. The wave function and the probability (the square norm in the strict sense) will be consequently studied.

- In this case we must emphasize the importance of the exponential factors that behave as $\propto e^{-n^2}$ which are suppression factors for large n and produce a rapid decay of the analyzed functions, that is to say, the system truly classicalizes.

- The other important point is the limit $\omega \to \sigma$ which is the degeneracy of the analytical functions describing the basic states of Mp(2): in the case of the entanglement probability P_{+-} for the even-odd sectors $(+-)$ it is an unique and fundamental projection leaving only a vestige of the circular angular variables as we will see explicitly in what follows:

B. Entanglement Probability P_{+-} for the crossed Even-Odd (+-) sectors

In this case the Mp(n) projected wave function takes the form

$$\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle = \frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{3/4}\sum_{n,m\,=\,0,1,2...}\frac{\left(\omega^*/2\right)^{2n}}{\sqrt{2n!}}\frac{\left(\sigma^*/2\right)^{2m}}{\sqrt{2m!}}e^{-4\left(n^2+m^2\right)}\times \tag{16}$$

$$\times\left[e^{2((l-i\varphi)n+(l'-i\varphi')m)}\frac{\left(\sigma^*\,e^{(l'-i\varphi')}/2\right)}{\sqrt{2m+1}}e^{-(2m+1/2)}+e^{i\rho}e^{2((l-i\varphi)m+(l'-i\varphi')n)}\frac{\left(\sigma^*\,e^{(l-i\varphi)}/2\right)}{\sqrt{2m+1}}e^{-(2m+1/2)}\right]$$

The square norm Probability P_{+-}

$$P_{+-} \equiv \left\|\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle\right\|^2 = \frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{3/2} \tag{17}$$

$$\sum_{n,m\,=\,0,1,2...}\frac{\left(|\omega|^2/4\right)^{2n}}{2n!}\frac{\left(|\sigma|^2/4\right)^{2m+1}}{(2m+1)!}e^{-4\left(n^2+m^2\right)}.$$

$$\cdot e^{-(4m+1)}\left[\left(e^{4(ln+l'(m+1/2))}+e^{4(l'n+l(m+1/2))}\right)+2\,e^{2(l+l')(n+m+1/2)}\cos\left(2\left(\Delta\left(m-n\right)-\frac{\rho+1}{2}\right)\right)\right]$$

Let us notice the low dependence on the angular degeneracies (contained in Δ) of the entanglement expressions for the different projections in this case.

We consider now the limit $\omega \to \sigma$:

$$\lim_{\omega\to\sigma}\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle \;=\; \lim_{\omega\to\sigma}\left\{\frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{3/4}\sum_{n,m=0,1,2\ldots}\frac{\left(\omega^*/2\right)^{2n}}{\sqrt{2n!}}\frac{\left(\sigma^*/2\right)^{2m+1}}{\sqrt{2m+1!}}\times\right. \tag{18}$$

$$\left.\times e^{-4\left(n^2+m^2\right)}e^{-(2m+1/2)}\left[e^{2((l-i\varphi)n+(l'-i\varphi')m)}\;e^{(l'-i\varphi')}\;+e^{i\rho}e^{2((l-i\varphi)m+(l'-i\varphi')n)}\;e^{(l-i\varphi)}\;\right]\right\}$$

The corresponding norm square Probability is:

$$\lim_{\omega\to\sigma}\left\|\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle\right\|^2= \tag{19}$$

$$=\lim_{\omega\to\sigma}\left\{\frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{3/2}\sum_{n,\,m\,=\,0,1,2\ldots}\frac{\left(|\omega|^2/4\right)^{2n}}{2n!}\frac{\left(|\sigma|^2/4\right)^{2m+1}}{(2m+1)!}\;e^{-4\left(n^2+m^2\right)}.\right.$$

$$\left.\cdot e^{-(4m+1)}\left[\left(e^{4(ln+l'(m+1/2))}+e^{4(l'n+l(m+1/2))}\right)\;+2\,e^{2(l+l')(n+m+1/2)}\cos\left(2\left(\Delta\,(m-n)-\frac{\rho+1}{2}\right)\right)\right]\right\}$$

$$=\sum_{n,\,m\,=\,0,1,2\ldots}\delta_{2n,2m+1}\;e^{-4\left(n^2+m^2\right)}.$$

$$e^{-(4m+1)}\left[\left(e^{4(ln+l'(m+1/2))}+e^{4(l'n+l(m+1/2))}\right)\;+2\;e^{2(l+l')(n+m+1/2)}\cos\left(2\;\left(\Delta\,(m-n)-\frac{\rho+1}{2}\right)\right)\right]$$

$$=\;\left[\,1+\cos\,(-\,(\,\Delta+\rho+1\,)\,)\,\right]\sum_{n\,=\,0,1,2\ldots}e^{-8\left(n^2+3/4\right)}\;2\;e^{4\,(\,l'+l\,)\,n}$$

When $(l'+l)=0$ the entanglement Probability reduces to :

$$\lim_{\omega\to\sigma}\;\left\|\left\langle \Psi^{(+)}(\omega)\,\Psi^{(-)}(\sigma)\right|\left|\Phi\right\rangle\right\|^2\;=\;\left(\frac{1+\vartheta_3\left(0,e^{-8}\right)}{e^6}\right)\left(1+\cos\left(\Delta+\rho+1\right)\right)$$

where $\vartheta_3\left(0,e^{-8}\right)$ is the respective Theta function. We can immediately see the dependence of the norm square on the angle variables (φ,φ') through $\Delta=(\varphi-\varphi')$ in the above limit.

C. Entanglement Probability $\mathbf{P}_{++}$ for the Even (++) sectors

In this case the wave function is

$$\left\langle \Psi^{(+)}(\omega)\,\Psi^{(+)}(\sigma)\right|\left|\Phi\right\rangle=\frac{1}{2}\left(1-|\omega|^2\right)^{1/4}\left(1-|\sigma|^2\right)^{1/4} \tag{20}$$

$$\sum_{n,\,m\,=\,0,1,2\ldots}\frac{\left(\omega^*/2\right)^{2n}}{\sqrt{2n!}}\frac{\left(\sigma^*/2\right)^{2m}}{\sqrt{2m!}}e^{-2\left(n^2+m^2\right)}\left[e^{2(ln+l'm-i(\varphi n+\varphi' m))}+e^{i\rho}e^{2(ln+l'm-i(\varphi n+\varphi' m))}\right]$$

And the norme square Probability is:

$$\left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(+)}(\sigma) \right| |\Phi\rangle \right\|^2 \;=\; \frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{1/2} \tag{21}$$

$$\sum_{n,\,m\,=\,0,1,2\,...} \frac{\left(|\omega|^2/4\right)^{2n}}{2n!}\frac{\left(|\sigma|^2/4\right)^{2m}}{2m!}\, e^{-4\left(n^2+m^2\right)} \times \tag{22}$$

$$\times \left[\, e^{4\,(ln+l'm)} + e^{\,4\,(lm+l'\,n)} + 2\, e^{2(\,l+l'\,)\,(m+n)} \cos\left(\,2\,\Delta\,\left(\,m-n\,\right)+\rho\,\right)\right]$$

We can see in all the cases the expressions for the entanglement Probabilities are similar.

For the limit $\omega \rightarrow \sigma$:

$$\lim_{\omega\rightarrow\sigma} \left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(+)}(\sigma) \right| |\Phi\rangle \right\|^2 = \frac{1}{2} \tag{23}$$

$$\sum_{n,\,m\,=\,0,1,2...} \delta_{n,m} e^{-4\left(n^2+m^2\right)} \left[e^{4(ln+l'm)} + e^{\,4\,(lm+l'n)} + 2\,e^{2\,(\,l+l'\,)(m+n)}\cos\,\left(\,2\,\Delta\,\left(m-n\right)+\rho\,\right)\right]$$

$$= \sum_{n\,=\,0,\,1,\,2...} e^{-8\,n^2+4\,n\,(\,l+l'\,)}\left(\,1+\cos\rho\,\right) \tag{24}$$

We can see immediately that there are **no dependence of the norm square entanglement Probability on the angle variables** (φ, φ') **through** Δ. Consequently, when $(l'+l)=0$, then :

$$\mathop{lim}_{\omega\rightarrow\sigma} \;\left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(+)}(\sigma) \right| |\Phi\rangle \right\|^2 \;=\; \left[\,1+\vartheta_3\left(0, e^{-8}\right)\,\right]\left(1+\cos\rho\right)$$

D. Entanglement Probability P_{--} for the Odd (- -) sectors

Finally, as before, the wave function is

$$\left\langle \Psi^{(-)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle = \frac{1}{2}\left(1-|\omega|^2\right)^{3/4}\left(1-|\sigma|^2\right)^{3/4} \tag{25}$$

$$\sum_{n,\,m\,=\,0,1,2...} \frac{\left(\omega^*/2\right)^{2n+1}}{\sqrt{2n+1!}}\frac{\left(\sigma^*/2\right)^{2m+1}}{\sqrt{2m+1!}} e^{-4\left(n^2+m^2\right)} e^{-2(m+n+1/2)} \times$$

$$\left[e^{2((l-i\varphi)(n+1)+(l'-i\varphi')(m+1))} \quad + e^{i\rho} e^{2((l-i\varphi)(m+1)+(l'-i\varphi')(n+1))}\right]$$

And the respective norm square is:

$$\left\| \left\langle \Psi^{(-)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle \right\|^2 = \frac{1}{2}\left(1-|\omega|^2\right)^{3/2}\left(1-|\sigma|^2\right)^{3/2} \tag{26}$$

$$\sum_{n,m=0,1,2...} \frac{\left(|\omega|^2/4\right)^{2n+1}}{2n+1!}\frac{\left(|\sigma|^2/4\right)^{2m+1}}{2m+1!} e^{-4\left(n^2+m^2\right)} e^{-4(m+n+1/2)} \times$$

$$\times \left\{ \left[e^{4(l(n+1/2)+l'(m+1/2))} + e^{4(l'(n+1/2)+l(m+1/2))} \right] + 2\, e^{2(l+l')(m+n+1)} \cos\left(2\,\left(\Delta(m-n)+\rho\right)\right) \right\}$$

The limit $\omega \to \sigma$ is :

$$\lim_{\omega\to\sigma} \left\| \left\langle \Psi^{(-)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle \right\|^2 = \frac{1}{2}\sum_{n,m=0,1,2...} \delta_{m,n} e^{-4\left(n^2+m^2\right)} e^{-4(m+n+1/2)} \times \tag{27}$$

$$\times \left\{ \left[e^{4(l(n+1/2)+l'(m+1/2))} + e^{4(l'(n+1/2)+l(m+1/2))} \right] + 2\, e^{2(l+l')(m+n+1)} \cos\left(2\,\left(\Delta(m-n)+\rho\right)\right) \right\} =$$

$$= \sum_{n=0,1,2...} e^{-8\left(n^2+n+1/4\right)}\, e^{4(l+l')(n+1/2)} \left[1 + \cos\left(2\,\rho\right)\right] \tag{28}$$

Again, as in the limit $\omega \to \sigma$ for the even (++) states the odd-odd (- -) Entanglament **is independent of the angular variables** (φ, φ') **through** Δ. When $l + l' = 0$ we have :

$$\lim_{\omega\to\sigma} P_{--} = \frac{1}{2}\vartheta_2\left(0, e^{-8}\right)\,\left(1 + \cos 2\rho\right)$$

Let us notice the low dependence on the angular degeneracies (contained in Δ) of the entanglement expressions for the different projection classicalizations in this case.

IV. ENTANGLEMENT OF THE COSET COHERENT STATES IN THE CIRCLE

A. Coset Coherent States in the Circle

For the sake of completeness, let us first summarize the steps to follow for the determination of the coset coherent states:

(i) The Coset G/H identification $\to E(2)/\mathbb{T}_2$, being $\mathbb{T}_2$ the group of translations $\{g_x, g_y\} \in \mathbb{T}_2$.

(ii) The Fiducial vector determination: It is annihilated by all the generators h of the stability subgroup H and for instance, invariant under the action of H. We propose: $|\, A_0 \,\rangle = A(z, x, y)\,|\varphi\rangle$ where $|\varphi\rangle$ is the London (circle) state, that is expanded in the $|n\rangle$ state of the harmonic oscillator and

$$A\,(z,x,y)_{(\pm)} \;=\; \left(e^{z} \;\pm\; e^{-z}\right) x \;+\; \left(\mp e^{z} \;+\; e^{-z}\right) y,$$

such that we can see: $\quad (e_x \;+\; e_y)\, A\,(z,x,y)_{(\,\pm)} \;=\; 0$

(iii) The coherent state is defined as the action of an element of the coset group on the fiducial vector $|A_0\rangle$, consequently the coherent state (still unnormalized yet) takes the form:

$$e^{-\alpha\,\partial_\varphi}\,|A_0\rangle == \frac{1}{\sqrt{2\pi}}\mathcal{S}\,(\alpha,\varphi)\underbrace{\sum_{n=0,1,2..} e^{-i(\varphi-\alpha/2)n}\,|n\rangle}_{|\varphi-\alpha/2\rangle}$$

$$= \frac{1}{\sqrt{2\pi}}\;\mathcal{S}\,(\alpha,\varphi)\;\;|\varphi-\alpha/2\;\rangle \tag{29}$$

where $\alpha \in \mathbb{C}$ is an arbitrary complex parameter in the element of the coset, which must be adjusted after normalization, being

$$\mathcal{S}\,(\alpha,\varphi)\;\equiv\;(A_{+}\cos\alpha + A_{-}\sin\alpha)$$

This is the most general coherent state from the Klauder-Perelomov construction which normalization takes the form

$$\langle\beta,\varphi'|\,|\alpha,\varphi\rangle \;=\; \frac{1}{2\pi}\;\frac{\mathcal{S}\,(\beta^{*},\varphi')\;\;\mathcal{S}\,(\alpha,\varphi)}{1-e^{-\,i\,(\,\varphi-\varphi'\,-\,(\alpha\,-\,\beta^{*})/2\,)}}$$

Then, the state is fully normalizable for $\varphi \rightarrow \varphi'$ iff the parameter α have $\mathrm{Im}\,\alpha \;\neq\; 0$:

$$|\,|\alpha,\varphi\rangle\,|^{2} \;=\; \frac{1}{2\pi}\;\frac{\mathcal{S}\,(\alpha^{*},\varphi)\;\;\mathcal{S}\,(\alpha,\varphi)}{1-e^{-\,i(\,\alpha^{*}\,-\,\alpha)/2}} \tag{30}$$

where:

$$\mathcal{S}\,(\alpha^{*},\varphi)\,\mathcal{S}\,(\alpha,\varphi) = \left(x^2+y^2\right)\cosh\left(2\,\mathrm{Im}\,\alpha\right) - \left(x^2-y^2\right)\sin 2\,(\mathrm{Re}\,\alpha-\varphi) + 2\,x\,y\cos 2\,(\mathrm{Re}\,\alpha-\varphi)$$

Consequently, it solves the problem of the London states that are overcomplete but clearly not normalizable when $\varphi \rightarrow \varphi'$:

$$\langle\varphi|\,|\varphi'\rangle \;=\; \frac{1}{2\pi}\sum_{n\,=\,0,1,2..} e^{i(\varphi-\varphi')n} \;=\; \frac{1}{2\pi}\frac{1}{1-e^{-i(\varphi-\varphi')}} \tag{31}$$

We see explicitely from these expressions Eq.(30), Eq.(31) how the general coherent states on the circle $|\alpha,\varphi\rangle$ (with the coherent characteristic complex parameter α) solve the problem of the non normalizability of the known (London, 't Hooft) $|\,\varphi\rangle$ states in the circle.

From Eq.(30) the normalized state coherent state $\langle\varphi|\,|\varphi'\rangle$ is:

$$|\alpha,\varphi\rangle \;=\; \underbrace{\sqrt{1-e^{-\operatorname{Im}\alpha}}\;e^{i\arg\mathcal{S}}}_{\mathcal{N}}\sum_{n=0,1,2..} e^{-i(\varphi-\alpha/2)\,n}\;|n\rangle \tag{32}$$

The identity resolved in a weak sense always for $\operatorname{Im}\alpha>0$ clearly showing the role played by the coherent state characteristic complex parameter α.

B. Action of the Mp(2) Group on the Coset States in the Circle

Again, let us look at the sector $s=1/4$ of the Hilbert space spanned by the $Mp(2)$ coherent states (unnormalized), the basic state is

$$\left|\Psi^{(+)}(\omega)\right\rangle = \left(1-|\omega|^2\right)^{1/4}\sum_{n=0,1,2..}\frac{(\omega/2)^{2n}}{\sqrt{2n!}}\;|\,2n\,\rangle$$

On the other hand:

$$\langle\alpha,\varphi| = \mathcal{N}^*\sum_{n=0,1,2..} e^{i(\varphi-\alpha^*/2)n}\;\langle\,n\,| \tag{33}$$

Therefore, we have

$$\begin{aligned}\langle\alpha,\varphi|\left|\Psi^{(+)}(\omega)\right\rangle &= \frac{\left(1-|\omega|^2\right)^{1/4}}{\sqrt{2\pi}}\sum_{n=0,1,2..}\frac{\left(\omega e^{i(\varphi-\alpha^*/2)}/2\right)^{2n}}{\sqrt{(2n)!}}\\ &= \frac{\left(1-|\omega|^2\right)^{1/4}}{\sqrt{2\pi}}\sum_{n=0,1,2..}\frac{(z'/2)^{2n}}{\sqrt{(2n)!}}\end{aligned} \tag{34}$$

where

$$\omega\,e^{i(\varphi-\alpha^*/2)} \;=\; z\,e^{-i\,\alpha^*/2} \;\equiv\; z',$$

and we see that the analytic function in the disc is now modified by the complex phase $(\,\varphi-\alpha^*/2\,)$.

Similarly, for the sector $s=3/4$ of the $Mp(2)$ states, that is the odd (-) states, we have:

$$\begin{aligned}\langle\alpha,\varphi|\left|\Psi^{(-)}(\omega)\right\rangle &= \frac{\left(1-|\omega|^2\right)^{3/4}}{\sqrt{2\pi}}\sum_{n=0,1,2..}\frac{\left(\omega e^{i(\varphi-\alpha^*/2)}/2\right)^{2n+1}}{\sqrt{(2n+1)!}}\\ &= \frac{\left(1-|\omega|^2\right)^{3/4}}{\sqrt{2\pi}}\sum_{n=0,1,2..}\frac{(z'/2)^{2n+1}}{\sqrt{(2n+1)!}}\end{aligned} \tag{35}$$

Notice that by taking the scalar product between the coset coherent state $|\alpha, \varphi\rangle$ and the $Mp(2)$ coherent states $\left|\Psi^{(-)}(\omega)\right\rangle$, we obtain two non-equivalent expansions in terms of analytical functions on the disk for the sectors of the minimal representations $s = 1/4$ and $s = 3/4$: **even and odd n states** respectively, in the eigenstaes $|\, n\,\rangle$ of the harmonic oscillator.

Consequently, $\left(\omega\, e^{i\,(\varphi - \alpha^*/2)} \;\equiv\; z'\,\right)$:

$$\langle\, \alpha, \varphi|\, \left|\Psi^{(\pm)}\,(z')\,\right\rangle = \begin{cases} \left(1 - |z'|^2\right)^{1/4} \sum\limits_{n\,=\,0,1,2..} \frac{(z'/2)^{2n}}{\sqrt{2n!}} & (+):\; \textit{even states} \\ \\ \left(1 - |z'|^2\right)^{3/4} \sum\limits_{n\,=\,0,1,2..} \frac{(z'/2)^{2n+1}}{\sqrt{(2n+1)!}} & (\text{-}):\; \textit{odd states} \end{cases} \tag{36a}$$

Therefore, for the total projected state, $\langle\, \alpha, \varphi\,|\,|\; \Psi\,(z')\;\rangle$:

$$\langle\, \alpha, \varphi|\,|\; \Psi\,(z')\;\rangle \;=\; \langle\, \alpha, \varphi|\,\big|\; \Psi^{(+)}\,(z')\;\big\rangle \;+\; \langle\, \alpha, \varphi|\,\big|\; \Psi^{(-)}\,(z')\;\big\rangle \;, \tag{36b}$$

We have

$$\langle\, \alpha, \varphi|\,|\Psi\,(z')\,\rangle = \left(1 - |z'|^2\right)^{1/4} \sum_{n\,=\,0,1,2..} \frac{(z'/2)^{2n}}{\sqrt{2n!}} \left[1 + \left(1 - |z'|^2\right)^{1/2} \frac{(z'/2)}{\sqrt{2n+1}}\right] \tag{37}$$

Let us notice the following observations:

(i) The analyticity condition of the function $\langle\alpha, \varphi|\,|\Psi\,(z')\rangle$ on the disk now constrained taking into account $|\, z'| \;=\; |\omega|\, e^{-\,\mathrm{Im}\,\alpha/2} \;<\; 1$ occurs under the already accepted condition arising from the normalization function.

(ii) The topology of the circle induced by the coset coherent state $|\alpha, \varphi\rangle$ Eq. (32) not only modifies the phase of ω (e.g: $\omega\, e^{i\,(\varphi - \alpha^*/2)} = z'$) but also the ratio of the disc due the displacement generated by the action of the coset.

(iii) The square norm of Eq.(37) is easily calculated giving as a result the function:

$$|\,\langle\alpha, \varphi|\,|\Psi\,(z')\rangle\,|^2 \;=\; \left(1 - |\,z'|^2\right)^{1/2} \cosh\left(\frac{|z'|^2}{2}\right) \;+\; \left(1 - |z'|^2\right)^{3/2} \sinh\left(\frac{|z'|^2}{2}\right) \;+$$
$$+\; \left(1 - |\,z'|^2\right)^{1/2} \;\mathrm{Re}\,(z') \sum_{n\;=\;0\,,1\,,2..} \frac{|z'/2|^{\,4n}}{2n\,!\;(2n+1)},$$

where $z' = \omega\, e^{i(\varphi - \alpha^*/2)}$. It shows a decreasing tail as n increases, and the analyticity, in this case in the disc $|z'| < 1$, with the same comments as in the items **(i)-(ii)** above.

Les us now consider the **Entanglement in this case**. We define now by analogy with the London states, the variables for the coset circle states in the Bargmann representation :

$$\begin{aligned}
z_1 &\equiv \omega\, e^{i(\varphi-\alpha^*/2)} &&\rightarrow \left\langle \varphi, \alpha \right| \left| \Psi^{(\pm)}(\omega) \right\rangle \equiv \Psi^{(\pm)}(z_1)\,, \\
z_1' &\equiv \omega\, e^{i(\varphi'-\alpha'^*/2)} &&\rightarrow \left\langle \varphi', \alpha' \right| \left| \Psi^{(\pm)}(\omega) \right\rangle \equiv \Psi^{(\pm)}(z_1') \\
z_2 &\equiv \sigma e^{i(\varphi-\alpha^*/2)} &&\rightarrow \left\langle \varphi, \alpha \right| \left| \Psi^{(\pm)}(\sigma) \right\rangle \equiv \Psi^{(\pm)}(z_2) \\
z_2' &\equiv \sigma e^{i(\varphi'-\alpha'^*/2)} &&\rightarrow \left\langle \varphi', \alpha' \right| \left| \Psi^{(\pm)}(\sigma) \right\rangle \equiv \Psi^{(\pm)}(z_2')
\end{aligned}$$

Then, the **Entanglement** expresses as:

$$\left\langle \Psi^{(\pm)}(\omega)\, \Psi^{(\pm)}(\sigma) \right| \left| \Phi \right\rangle = \frac{1}{2} \left(\Psi^{(\pm)}(z_1) \otimes \Psi^{(\pm)}(z_2') + e^{i\rho} \Psi^{(\pm)}(z_1') \otimes \Psi^{(\pm)}(z_2) \right)$$

Explicitly, for the completely projected states, namely, **the full entangled wave function**, we have:

$$\begin{aligned}
\left\langle \Psi(\omega)\, \Psi(\sigma) \right| \left| \Phi \right\rangle &= \frac{1}{2} Z_1^{1/4} Z_2'^{1/4} \sum_{n,m=1,2...} \frac{\left(z_1^*/2\right)^{2n}}{\sqrt{2n!}} \frac{\left(z_2'^*/2\right)^{2m}}{\sqrt{2m!}} \times \\
&\left(1 + Z_1^{1/2} \frac{(z_1^*/2)}{\sqrt{2n+1}}\right) \left(1 + Z_2'^{1/2} \frac{(z_2'^*/2)}{\sqrt{2m+1}}\right) + \frac{1}{2} Z_1'^{1/4} Z_2^{1/4} \sum_{n,m=1,2...} \frac{\left(z_1'^*/2\right)^{2n}}{\sqrt{2n!}} \frac{\left(z_2^*/2\right)^{2m}}{\sqrt{2m!}} \times \\
&e^{i\rho} \left(1 + Z_1'^{1/2} \frac{(z_1'^*/2)}{\sqrt{2n+1}}\right) \left(1 + Z_2^{1/2} \frac{(z_2^*/2)}{\sqrt{2m+1}}\right)
\end{aligned} \tag{38}$$

where the following convenient notation have been introduced:

$$Z_1 \equiv 1 - |z_1|^2, \quad Z_1' \equiv 1 - |z_1'|^2, \quad Z_2 \equiv 1 - |z_2|^2, \quad Z_2' \equiv 1 - |z_2'|^2 \tag{39}$$

And the **full Entanglement Probability** takes the following form:

$$\begin{aligned}
\left\| \left\langle \Psi(\omega)\, \Psi(\sigma) \right| \left| \Phi \right\rangle \right\|^2 &= \left\{ \frac{1}{2} Z_1^{1/4} Z_2'^{1/4} \sum_{n,m=1,2...} \frac{\left(z_1^*/2\right)^{2n}}{\sqrt{2n!}} \frac{\left(z_2'^*/2\right)^{2m}}{\sqrt{2m!}} \times \right. \\
&\left(1 + Z_1^{1/2} \frac{(z_1^*/2)}{\sqrt{2n+1}}\right) \left(1 + Z_2'^{1/2} \frac{(z_2'^*/2)}{\sqrt{2m+1}}\right) + \frac{1}{2} Z_1'^{1/4} Z_2^{1/4} \sum_{n,m=1,2...} \frac{\left(z_1'^*/2\right)^{2n}}{\sqrt{2n!}} \frac{\left(z_2^*/2\right)^{2m}}{\sqrt{2m!}} \times \\
&e^{i\rho} \left(1 + Z_1'^{1/2} \frac{(z_1'^*/2)}{\sqrt{2n+1}}\right) \left(1 + Z_2^{1/2} \frac{(z_2^*/2)}{\sqrt{2m+1}}\right) \times \frac{1}{2} Z_1^{1/4} Z_2'^{1/4} \sum_{n,m=1,2...} \frac{\left(z_1/2\right)^{2n}}{\sqrt{2n!}} \frac{\left(z_2'/2\right)^{2m}}{\sqrt{2m!}} \times \\
&\left(1 + Z_1^{1/2} \frac{(z_1/2)}{\sqrt{2n+1}}\right) \left(1 + Z_2'^{1/2} \frac{(z_2'/2)}{\sqrt{2m+1}}\right) + \\
&\left. + \frac{1}{2} Z_1'^{1/4} Z_2^{1/4} \sum_{n,m=1,2...} \frac{\left(z_1'/2\right)^{2n}}{\sqrt{2n!}} \frac{\left(z_2/2\right)^{2m}}{\sqrt{2m!}} \times e^{-i\rho} \left(1 + Z_1'^{1/2} \frac{(z_1'/2)}{\sqrt{2n+1}}\right) \left(1 + Z_2^{1/2} \frac{(z_2/2)}{\sqrt{2m+1}}\right) \right\}
\end{aligned}$$

C. Entanglement and the Minimal Group Representation for the Coset States

In this case, for the circle coset coherent states, the unique and fundamental projection due to the underlying metaplectic structure is given by the same analytical functions $\omega \equiv \sigma$, and $(z_1, z_1') \equiv (z_2, z_2')$, being the respective wave function :

$$\left\langle \Psi^{(+)}(\omega)\, \Psi^{(-)}(\omega) \right| |\Phi\rangle \;=\; \frac{1}{2}\, Z_1^{1/4}\, Z_1'^{\,3/4} \sum_{n,\,m\,=\,1,2\,...} \frac{(z_1^*/2)^{2n}}{\sqrt{(2n)\,!}} \frac{(z_1'^{\,*}/2)^{2m+1}}{\sqrt{(2m+1)\,!}} + \qquad (40)$$

$$+\; \frac{1}{2}\, Z_1'^{\,1/4}\, Z_1^{3/4}\, e^{i\rho} \sum_{n,\,m\,=\,1,2\,...} \frac{(z_1'^{\,*}/2)^{2n}}{\sqrt{2n\,!}} \frac{(z_1^*/2)^{2m+1}}{\sqrt{(2m+1)\,!}}$$

And the corresponding square norm **Entanglement Probability** is:

$$\left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(-)}(\omega) \right| |\Phi\rangle \right\|^2 = \qquad (41)$$

$$Z_1'^{\,3/2}\, Z_1^{1/2} \cosh\left(|z_1|^2/4\right)\, \sinh\left(|z_2'|^2/4\right) + Z_1'^{\,1/2}\, Z_1^{3/2} \cosh\left(|z_1'|^2/4\right)\, \sinh\left(|z_1|^2/4\right) +$$

$$+\, Z_1'\, Z_1 \left[e^{-i\rho} \cosh\left(z_1^* z_1'/4\right)\, \sinh\left(z_1'^* z_1/4\right) \,+\, e^{i\rho} \cosh\left(z_1 z_1'^*/4\right)\, \sinh\left(z_1' z_1^*/4\right) \right]$$

We can notice that the above expression contains trigonometric factors because:

$$z_1^*\, z_1' = |\omega|^2\, e^{-i\,\Delta}\, e^{i\,(\alpha-\alpha'^*)}, \qquad\qquad z_1^{*\prime}\, z_1 = |\omega|^2\, e^{i\Delta}\, e^{-i(\alpha-\alpha'^*)}$$

Therefore, e.g. $\cosh\left(z_1^* z_1'/4\right) = \cosh\left(|\omega|^2\, e^{-i\Delta} e^{i(\alpha-\alpha'^*)}/4\right)$, it could be expanded if one specifically knows the exponent $(\alpha-\alpha'^*)$.

As in the previous cases, we can see now the particular projections for the wave functions and the corresponding norm square probabilities:

D. Entanglement Probabilities

- Entanglement Probability $\mathbf{P}_{++}$ of the Even (++) sectors:

$$\left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(+)}(\sigma) \right| |\Phi\rangle \right\|^2 = (Z_1\, Z_2)^{1/2}/4 \qquad (42)$$

$$\left[\left(\frac{Z_1'}{Z_1}\right)^{1/2} \cosh\left(|z_1'|^2/4\right) \cosh\left(|z_2|^2/4\right) + \left(\frac{Z_2'}{Z_2}\right)^{1/2} \cosh\left(|z_1|^2/4\right)\, \cosh\left(|z_2'|^2/4\right) \right] +$$

$$+ \left(\frac{Z_1'\, Z_2'}{Z_1\, Z_2}\right)^{1/4} \left[e^{-i\rho} \cosh\left(z_1^* z_1'/4\right)\, \cosh\left(z_2'^* z_2/4\right) + e^{i\rho} \cosh\left(z_1 z_1'^*/4\right)\, \cosh\left(z_2' z_2^*/4\,\right) \right] \qquad (43)$$

- Entanglement Probability P_{+-} of the Even-Odd (+-) sectors:

$$\left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle \right\|^2 = \left(Z_1^{1/2}\, Z_2^{3/2}/4 \right)$$

$$\left\{ \left[\left(\frac{Z_1'}{Z_1} \right)^{1/2} \cosh\left(|z_1'|^2/4 \right) \sinh\left(|z_2|^2/4 \right) + \left(\frac{Z_2'}{Z_2} \right)^{3/2} \cosh\left(|z_1|^2/4 \right) \sinh\left(|z_2'|^2/4 \right) \right] + \right.$$

$$\left. + \left(\frac{Z_1'}{Z_1} \right)^{1/4} \left(\frac{Z_2'}{Z_2} \right)^{3/4} \left[e^{-i\rho} \cosh\left(z_1^* z_1'/4 \right) \sinh\left(z_2'^* z_2/4 \right) + e^{i\rho} \cosh\left(z_1 z_1'^*/4 \right) \sinh\left(z_2' z_2^*/4 \right) \right] \right\}$$

- Entanglement Probability P_{--} of the Even-Even (−) sectors:

$$\left\| \left\langle \Psi^{(-)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle \right\|^2 = \left(Z_1\, Z_2^{3/2}/4 \right) \tag{44}$$

$$\left\{ \left[\left(\frac{Z_1'}{Z_1} \right)^{3/2} \sinh\left(|z_1'|^2/4 \right) \sinh\left(|z_2|^2/4 \right) + \left(\frac{Z_2'}{Z_2} \right)^{3/2} \sinh\left(|z_1|^2/4 \right) \sinh\left(|z_2'|^2/4 \right) \right] + \right. \tag{45}$$

$$\left. + \left(\frac{Z_1'\, Z_2'}{Z_1\, Z_2} \right)^{3/4} \left[e^{-i\rho} \sinh\left(z_1^* z_1'/4 \right) \sinh\left(z_2'^* z_2/4 \right) + e^{i\rho} \sinh\left(z_1 z_1'^*/4 \right) \sinh\left(z_2' z_2^*/4 \right) \right] \right\} \tag{46}$$

Here we can see that being (σ, ω) and (φ, φ') involved in only one single variable z: (z_1, z_2, z_1', z_2'), the classicalization is more evident and the limits can be computed as in the previous cases. Recall that $Z : (Z_1, Z_2, Z_1', Z_2')$ just relates to $|z|^2$ Eq.(39): $Z_i = (1 - |z_i|^2)$ for each $i = (1, 2)$.

V. COMPARISONS OF ENTANGLEMENTS: SCHRODINGER CAT STATES AND MP(2) STATES

States: Let us first recall the correspondence between the Schrodinger cat states and the basic (+) and (-) $Mp(2)$ states, namely $|1/4\rangle$ and $|3/4\rangle$:

$Heisenberg - Weyl$		$Metaplectic\ Mp(2)$	
$\|\alpha_+\rangle$	$\longrightarrow$	$\|1/4\rangle$	Even
$\|\alpha_-\rangle$	$\longrightarrow$	$\|3/4\rangle$	Odd

(47)

where explicitly the standard cat Schrodinger states are

$$|\alpha_\pm\rangle \;=\; \frac{1}{\sqrt{2 \;\pm\; 2\,e^{-2|\alpha|^2}}}\;[\,|\,\alpha\,\rangle \;\pm\; |-\alpha\,\rangle\,]$$

which simply describe standard coherent states (e.g., Heisenberg-Weyl), α being the typical complex displacement parameter. From the density matrix point of view, the $Mp(2)$ states are favored with respect to the cat states because of the principle of minimum representation: The even and odd **Mp(2) states are irreducible representations while the even and odd cat states are not**. In the case of the standard cat states, the density matrices for the even and odd sectors are:

$$\rho_{\pm\alpha} \;=\; \frac{1}{2\;(\;1\;\pm\;e^{\,-2\,|\,\alpha\,|^{\,2}}\;)}\;[\,|\,\alpha\,\rangle\,\langle\,\alpha\,|\,+\,|-\alpha\,\rangle\,\langle\,-\alpha\,|\,\pm\,(\,|-\alpha\,\rangle\,\langle\,\alpha\,|\,+\,|\,\alpha\,\rangle\,\langle\,-\alpha\,|\,)\,] \tag{48}$$

while in the fundamental case of the minimal group representation $Mp(2)$, we have a *diagonal representation* clearly differentiated into the corresponding *even* and *odd* subspaces:

$$\rho_{Mp(2)} \;=\; \begin{cases} |\;1/4\;\rangle\,\langle\;1/4\;| \;\;\rightarrow\;\; \textit{even states} \\ |\;3/4\;\rangle\,\langle\;3/4\;| \;\;\rightarrow\;\; \textit{odd states} \end{cases} \tag{49}$$

The minimal group representation does also manifest in the fact that for the cat states $\rho_{\pm\alpha}$ have a tail while such tail does not appear in the fundamental expressions of $\rho_{Mp(2)}$ for both $Mp\,(2)$ even and odd sectors. The minimal Mp(2) representation **is diagonal : classical and quantum descriptions are both represented**, and describe **both continuum and discrete sectors**. (And this is particularly relevant in describing the fundamental *quantum substrate* of space time as we showed in our recent work Refs [11], [10], [7]. Moreover, the basic $Mp(2)$ even and odd states $|1/4\rangle\,,|3/4\rangle$ do not require any extrinsic generation process as the cat states, and can do form a generalized state of the type:

$$\left|\Psi_{Mp(2)}\right\rangle_{gen} \;=\; \frac{1}{\sqrt{|A|^2\;\pm\;|B\,|^2}}\;[\;A\;|\,1/4\,\rangle\;\pm\;B\;|\,3/4\,\rangle\;]$$

(similar to the standard Schrodinger cat state $|\alpha_\pm\rangle$), and giving in this case the following density matrix:

$$\rho_{Mp(2)_{gen}} = \frac{1}{|A|^2 \pm |B|^2}\left[\,|A|^2\;\;|1/4\rangle\,\langle 1/4|\;+\;|B|^2\;\;|3/4\rangle\,\langle 3/4|\;\pm\;(B^*A\;|3/4\rangle\,\langle 1/4| + A^*B\,|1/4\rangle\,\langle 3/4|)\,\right] \tag{50}$$

with coefficients A and B which can be fully computed depending on the problem considered.

Entanglements: With respect to the entanglement states in Ref [7], Schrodinger cat states are

$$|\alpha_+\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{(2n)!}} \; |2n\rangle$$

$$|\alpha-\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{(2n+1)!}} \; |2n+1\rangle$$

where the standard displacement operators $\mathcal{D}(\alpha)$ and $\mathcal{D}(-\alpha)$ have been combined operating on the fiducial of the harmonic oscillator, namely $|0\rangle$, to sweep out *even and odd* n-states. The **cat representation is not irreducible**, as is the case of the $Mp(n)$ basic states, which do span out irreducible minimal subspaces.

The wave functions (projections) of the Schrodinger cat states on the London states are:

$$\langle\varphi|\,|\alpha_+\rangle = \frac{e^{-\frac{1}{2}|\widetilde{\alpha}|^2}}{2\pi} \sum_{n=0}^{\infty} \frac{\widetilde{\alpha}^{2n}}{\sqrt{(2n)!}}$$

$$\langle\varphi|\,|\alpha_-\rangle = \frac{e^{-\frac{1}{2}|\widetilde{\alpha}|^2}}{2\pi} \sum_{n=0}^{\infty} \frac{\widetilde{\alpha}^{2n+1}}{\sqrt{(2n+1)!}}$$

where $\widetilde{\alpha} = \alpha\, e^{i\varphi}$. Therefore, the total or complete projected cat state on the circle is given by:

$$\langle\varphi|\,|\alpha_+\rangle \; + \; \langle\varphi|\,|\alpha_-\rangle \; = \; \frac{e^{-\frac{1}{2}|\alpha|^2}}{2\pi} \sum_{n=0}^{\infty} \frac{\widetilde{\alpha}^{n}}{\sqrt{n!}} \tag{51}$$

which is just the projected London state φ with the Heisenberg-Weyl standard coherent state $|\,\alpha\,\rangle$ and the complex parameter α. **This contrasts with the case of projecting the $Mp(2)$ states** where the complete projected state on the circle is given by:

$$\langle\, \varphi\,|\,|\, \Psi\,(\omega)\,\rangle \;=\; \frac{\left(1-|z|^2\right)^{1/4}}{\sqrt{2\pi}} \sum_{n=0,1,2..} \frac{(z/2)^{2n}}{\sqrt{(2n)\,!}} \left[1 \;+\; \left(1-|z|^2\right)^{1/2} \frac{(z/2)}{\sqrt{2n+1}} \right] \tag{52}$$

where we see the weight function $\left(1-|z|^2\right)^{1/4}$ indicating the geometry of the minimum irreducible subspace (in the even (+) treated here) and $0 \leq |z| < 1$ (the unitary disc). Recall too that the cat even and odd n representations are **not** irreducible.

- **In Figure 4** we show the Entanglement Probability P^{cat}_{+-} for the circle (London) states φ projected via the Schrodinger cat states for the orthogonal states $\varphi = \varphi' + \pi/2$: $\Delta \to \pi/2$. **In Figure 5** we reproduce the Entanglement Probability $P_{+\,-}$ for these circle (London) states φ but projected via the $Mp(2)$ states: Irreducible sectors of the minimal Hilbert space yielding **Classicalization**. By comparing both cases we can see the geometrical difference between the Entanglement probabilities: P^{cat}_{+-} projecting with the Schrodinger cat states versus P_{+-} projecting with the basic $Mp(2)$ states.

- **Cat Antipodal Entanglements: Figure 4** shows P^{cat}_{+-} (left side) and the **Antipodal Entanglement** P^{cat}_{A+-} of the circle London states for $\Delta \to \pi/2$, ie orthogonal states $\varphi = \varphi' + \pi/2$. The variables α and β in the Figure are in the range $0 \leq |\alpha|\,,\ |\beta| < 2$ to appreciate the shape of the Entanglement probability.

- **Mp(2) Antipodal Entanglements:** P_{+-} (left side) and the **Antipodal Entanglement** P_{A+-} of the circle London orthogonal states $\Delta \to \pi/2$ are shown in **Figure 4**. The variables ω and σ of the analytic functions in the unitary disk (which square norms are displayed in the Figure) being in the range : $0 \leq |\omega|\,,\ |\sigma| < 1$.

VI. CONCLUDING REMARKS

In this paper we have computed and analyzed the Entanglement of quantum physics within the new framework of our recent work Ref [6] on Classicalization. Thus, the results of our paper here are twofold : **Entanglement and Classicalization and the relationship between them**. In Section I we summarized the main results of the study performed here and we do not summarize them here again. In this Section we just briefly highlight main concluding remarks in a synthetic way. We consider various types of states on the circle : London states, new coherent coset states and states with cylinder topology as eigenstates of a lowering operator.

We highlight here the following conclusions of this paper:

(1) The Entanglement study performed here clearly supports the fact the metaplectic group naturally classicalizes any quantum state as the realization of the Minimal Group

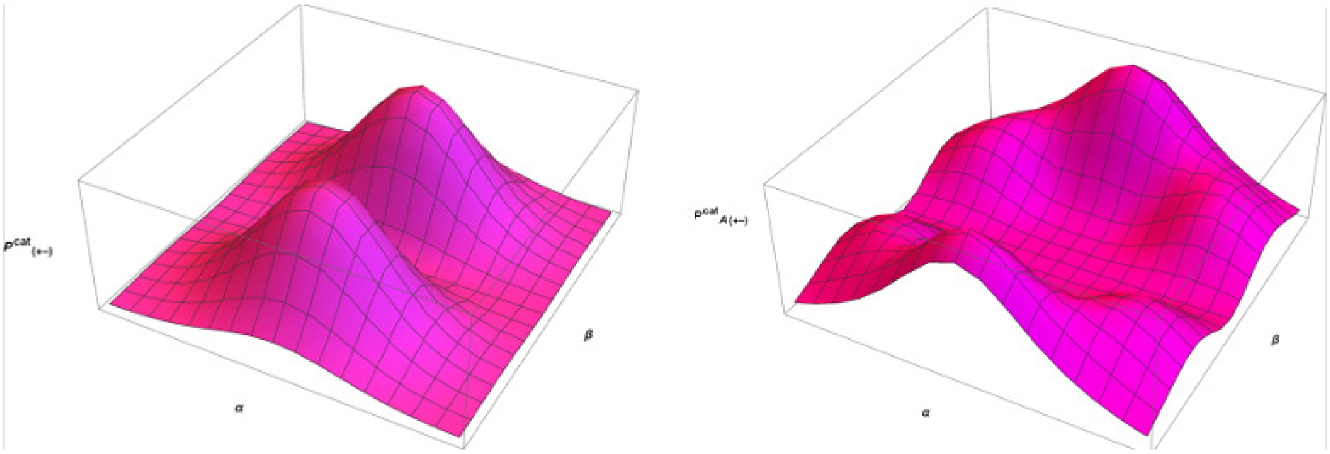

FIG. 4: **Schrodinger cat and circle Entanglement Probability of even and odd states:** $P^{cat}(+-)$ **(left side) and the Antipodal one** $P_A^{cat}(+-)$ **(right side) of the circle entangled orthogonal states** $\varphi = \varphi' + \pi/2 : \Delta \to \pi/2$ projected onto the Schrodinger cat even and odd states. The parameters α and β in the Figure represent the respective norms, being the range of the parameters $0 \le |\alpha|$ and $|\beta| < 2$ to appreciate the shape of the Entanglement Probability.

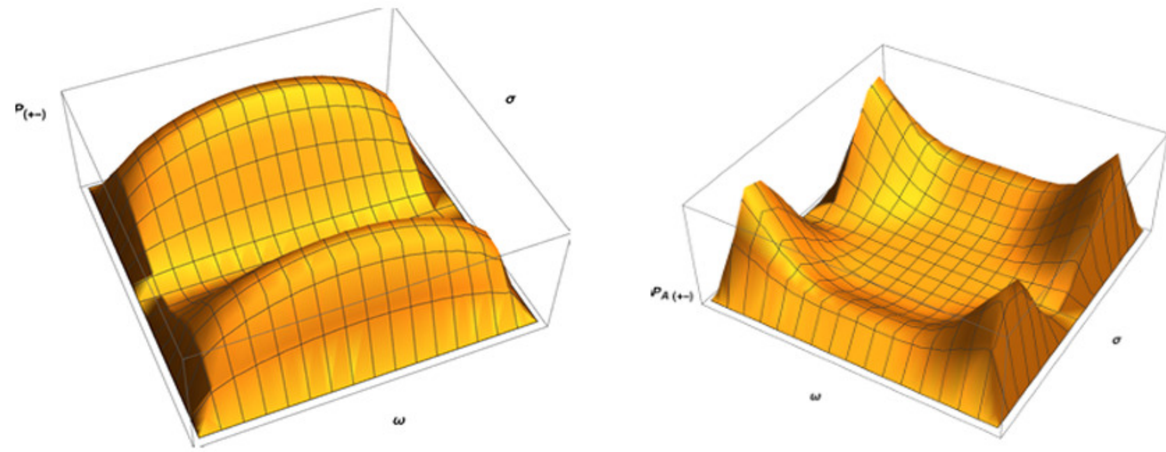

FIG. 5: **Mp(2) and circle Entanglement Probability of even and odd states:** $P(+-)$ **(left side) and the Antipodal one** $P_A(+-)$ **(right side)** of the circle orthogonal states $\varphi = \varphi' + \pi/2, (\Delta \to \pi/2)$, **projected on the basic Mp(2) states: Classicalization**. The variables ω and σ in the Figure representing the respective norms of the analytic functions in the unitary disk, have the range $0 \le |\omega|$ and $|\sigma| < 1$.

Representation Principle: this is uniquely realized by the Metaplectic group with its associated minimal Hilbert Space sectors, and the Entanglements of the different states show too **Minimal Entanglements, in particular zero** when projecting on the Mp(2) states: **Classicalization**).

(2) The result of the item (1) provides a precise conceptual and computational support to the meaning of classicalization with respect to the condition of taking a limit for $\hbar$ (or any other control parameter) for the classical description which is more a formal limit rather than a full conceptual description for classicalization.

(3) The projected Mp(2) **Entanglement Probability** is dependent on the angular variables, and through the difference $\Delta = (\varphi - \varphi')$ in all cases treated here except for the cylindrical coherent states. In the **cylindrical topology case** the entanglement becomes **independent of the angular variables** (observable) in the limit of coincident states ($\omega \to \sigma$), projected on a single subspace (even or odd) of the total Hilbert space. This is a **quantum memory loss** manifestation in the Robertson sense, that is, there is no decoherence in the sense of interaction with an environment.

(4) Entanglement Classicalization is **stronger** for the cylinder topology than for the circle: Decay of the Entanglement Probability for large n (even and odd levels $n = 1, 2, 3, ...$) is decreasing exponentially with $(2n)^2$ and $(2n+1)^2$ in the cylinder, while it is decreasing exponentially with $(2n)$ and $(2n+1)$ in the circle.

(5) Comparison of the Entanglements projected on the Mp(2) states and on other (not Mp(2)) states clearly shows the differences between the two cases **both for the Entanglement and Classicalization. Figures 4 and 5** exhibit the differences between the Schrodinger cat projected Entanglement Probabilities and the Mp(2) Entanglement Probabilities.

(6) Comparison of the **Antipodal Entanglement** (regulated by the phase control parameter $\rho = \pi$) with the Non Antipodal ($\rho = 0$) one reveals interesting properties in all cases studied here, for example (as in **Figure 5**) the **low and Minimal Entanglement** occurring in the Mp(2) projected states, as another manifestation of **Classicalization**, particularly in the **Antipodal Entanglement**.

(7) Outlook and Implications: The results of this paper on theoretical and conceptual aspects of quantum theory and its classicalization can have impact and applications in different branches of physics and other disciplines, classical and quantum information processing, quantum computation and experimental research, the quantum-classical interaction and interpretation measurements, the classical-quantum duality, the classical or quantum optimization. **For instance:**

(i) By choosing a type of geometry- topology of the states, one can obtain stronger or lower classicalization.

(ii) By using or avoiding coincident or orthogonal states, one can allows or avoid that entanglement be decreasing, broken or vanishing.

(iii) By choosing Antipodal or Non Antipodal Entanglement (regulating the phase control parameter ρ be equal to π or 0 respectively), Entanglement can be lower, minimal or not.

(iv) Combination of possibilities **(i)-(ii)-(iii)** could yield more effects.

VII. ACKNOWLEDGEMENTS

D.J.C-L acknowledges the Special Astrophysical Observatory of the Russian Academy of Sciences and CONICET of Argentine for Institutional and financial support.
N.G.S acknowledges useful communications with Gerard 't Hooft and Jose Luis Mac Loughlin on various occasions.

VIII. APPENDIX I: MP(2) PROJECTED ENTANGLEMENT PROBABILITIES IN THE CIRCLE

Below we provide now the explicit expressions of the square norm Entanglement Probabilities P and their limits $\omega \to \sigma$ (analytic degeneracy) for the circle (London type) states.It is convenient to introduce the following notation:

$$\beta = \frac{|\omega|^2}{4} \cos \Delta, \qquad \widetilde{\beta} = \frac{|\omega|^2}{4} \sin \Delta \tag{53}$$

$$\gamma = \frac{|\sigma|^2}{4} \cos \Delta, \qquad \widetilde{\gamma} = \frac{|\sigma|^2}{4} \sin \Delta \tag{54}$$

(1) The Even-even sector Probability P_{++} is given by:

$$P_{++} \equiv \left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(+)}(\sigma) \right| |\Phi\rangle \right\|^2 = \frac{1}{2}\sqrt{\left(1-|\omega|^2\right)\left(1-|\sigma|^2\right)} \Big\{ \cosh\beta \cosh\widetilde{\beta} +$$
$$+ \cos\rho \left[\cosh\beta \cos\widetilde{\beta} \cosh\gamma \cos\widetilde{\gamma} + \sinh\beta \sin\widetilde{\beta} \sinh\sigma \sin\widetilde{\sigma} \right] +$$
$$+ \sin\rho \left[\sinh\beta \sin\widetilde{\beta} \cosh\gamma \cos\widetilde{\gamma} - \sinh\gamma \sin\widetilde{\gamma} \cosh\beta \cos\widetilde{\beta} \right] \Big\}$$

Limit $\omega \to \sigma$:

$$\lim_{\omega \to \sigma} P_{++} = \frac{1}{2}\left(1-|\omega|^2\right) \times$$
$$\left\{ \cosh^2\left(\frac{|\omega|^2}{4}\right) + \cos\rho \left[\cosh^2\left(\frac{|\omega|^2}{4}\cos\Delta\right) - \sin^2\left(\frac{|\omega|^2}{4}\sin\Delta\right) \right] \right\}$$

(2) Even-odd sector Probability P_{+-} **:**

$$P_{+-} \equiv \left\| \left\langle \Psi^{(+)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle \right\|^2 = \frac{1}{2}\left(1-|\omega|^2\right)^{1/2}\left(1-|\sigma|^2\right)^{3/2} \left\{ \cosh\left(\frac{|\omega|^2}{4}\right) \sinh\left(\frac{|\sigma|^2}{4}\right) + \right.$$
$$+ \left[\cosh\beta \cos\widetilde{\beta} \sinh\gamma \cos\widetilde{\gamma} + \sinh\beta \sin\widetilde{\beta} \cosh\gamma \sin\widetilde{\gamma} \right] \cos\rho +$$
$$\left. + \left[\cosh\beta \cos\widetilde{\beta} \cosh\gamma \sin\widetilde{\gamma} - \sinh\beta \sin\widetilde{\beta} \sinh\gamma \cos\widetilde{\gamma} \right] \sin\rho \right\}$$

Limit $\omega \to \sigma$:

$$\lim_{\omega \to \sigma} P_{+-} = \frac{1}{2}\left(1-|\omega|^2\right)^2 \left\{ \cosh\left(\frac{|\omega|^2}{4}\right) \sinh\left(\frac{|\omega|^2}{4}\right) + \right.$$
$$\left. + \cosh\left(\frac{|\omega|^2}{4}\cos\Delta\right) \sinh\left(\frac{|\omega|^2}{4}\cos\Delta\right) \cos\rho + \cos\left(\frac{|\omega|^2}{4}\sin\Delta\right) \sin\left(\frac{|\omega|^2}{4}\sin\Delta\right) \sin\rho \right\}$$

(3) Odd-odd sector Probability P_{--} **:**

$$P_{--} \equiv \left\| \left\langle \Psi^{(-)}(\omega)\, \Psi^{(-)}(\sigma) \right| |\Phi\rangle \right\|^2 = \frac{1}{2}\left[\left(1-|\omega|^2\right)\left(1-|\sigma|^2\right)\right]^{3/2} \left\{ \sinh\left(\frac{|\omega|^2}{4}\right) \sinh\left(\frac{|\sigma|^2}{4}\right) + \right.$$
$$+ \left[\sinh\beta \cos\widetilde{\beta} \sinh\gamma \cos\widetilde{\gamma} + \cosh\beta \sin\widetilde{\beta} \cosh\gamma \sin\widetilde{\gamma} \right] \cos\rho +$$
$$\left. + \left[\sinh\beta \cos\widetilde{\beta} \cosh\gamma \sin\widetilde{\gamma} - \cosh\beta \sin\widetilde{\beta} \sinh\gamma \sin\widetilde{\gamma} \right] \sin\rho \right\}$$

Limit $\omega \to \sigma$:

$$\lim_{\omega \to \sigma} P_{--} \;=\; \frac{1}{2}\left(1-|\omega|^2\right)^3 \;\times$$
$$\left\{ \sinh^2\left(\frac{|\omega|^2}{4}\right) + \left[\sinh^2\left(\frac{|\omega|^2}{4}\cos\Delta\right) + \sin^2\left(\frac{|\omega|^2}{4}\sin\Delta\right)\right]\cos\rho \right\}$$

As we see here, for the entangled Mp(2)-projected London states, the limits $\omega \to \sigma$ (analytic degeneracy) are (φ, φ') dependent through $\Delta \equiv (\varphi - \varphi')$ in all cases.

---

[1] Ortwin Hess *From Quantum physics to Quantum science*, APL Quantum 1, 010401 (2024); https://doi.org/10.1063/5.0202749

[2] Alain Aspect, John F. Clauser and Anton Zeilinger, Nobel Prize in Physics 2022, "for experiments with entangled photons, ... and pioneering quantum information science" https://www.nobelprize.org/uploads/2023/10/advanced-physicsprize2022-4.pdf
and Refs therein.

[3] John J. Hopfield and Geoffrey E. Hinton, Nobel Prize in Physics 2024, "for foundational discoveries and inventions that enable machine learning with artificial neural networks" https://www.nobelprize.org/uploads/2024/10/advanced-physicsprize2024-2.pdf
and Refs therein.

[4] J. L. Mac Loughlin, N. G. Sanchez, "Photography as a New Conceptual Quantum Information System", (Oct. 2023)
https://www.researchgate.net/publication/375745724
https://www.bibsonomy.org/bibtex/7a77b52e38f27089f47229b1193dc005

[5] The International Year of Quantum Science and Technology (IYQ-2025),
See for example: https://quantum2025.org/
$https : //qt.eu/news/2024/2024 - 06 - 07_UN - declares - 2025 - international - year -$

$of - quantum - science - and - technology$

[6] D. J. Cirilo-Lombardo, N. G. Sanchez, APL Quantum 2, 016104 (2025). https://doi.org/10.1063/5.0247698 https://pubs.aip.org/aip/apq/article/2/1/016104/3329660/Classical-ontological-dual-states-in-quantum

[7] D. J. Cirilo-Lombardo and N. G. Sanchez, Entanglement and Generalized Berry Geometrical Phases in Quantum Gravity, *Symmetry*, 16(8), 1026 (2024) https://doi.org/10.3390/sym16081026

[8] G. 't Hooft, *The Hidden Ontological Variable in Quantum Harmonic Oscillators* , Front. Quantum Sci. Technol. 3, 1 (2024). https://doi.org/10.3389/frqst.2024.1505593

[9] F. London, *Winkelvariable und Kanonische Transformationen in der Undu- lationsmechanik: Zeitschrift fur Physik*, 37, 915-925, (1926). https://doi.org/10.1007/bf01397484

[10] D. J. Cirilo-Lombardo, N. G. Sanchez, Universe, 10, 22 (2024). https://doi.org/10.3390/universe10010022

[11] D. J. Cirilo-Lombardo, N. G. Sanchez, Phys. Rev.D 108, 126001 (2023). https://doi.org/10.1103/PhysRevD.108.126001

[12] N. G. Sanchez, Phys. Rev. D 104, 123517 (2021). https://doi.org/10.1103/physrevd.104.123517

[13] N. G. Sanchez, Phys. Rev. D 107, 126018 (2023). https://doi.org/10.1103/physrevd.107.126018

[14] N. G. Sanchez, Int. J. Mod Phys D28, 1950055 (2019).
https://doi.org/10.1142/S021827181950055X

[15] N. G. Sanchez, Int. J. Mod Phys A34, 1950155 (2019).
https://doi.org/10.1142/S0217751X19501550

[16] N. G. Sanchez, Gravit. Cosmol. 25, 91-102 (2019).
https://doi.org/10.1134/S0202289319020142

[17] D. J. Cirilo-Lombardo, J. Math. Phys. 57 (2016) 6, 063503.
https://doi.org/10.1007/s10701-007-9133-9

[18] D. J. Cirilo-Lombardo; Found. Phys. 37 (2007) 919-950.
https://doi.org/10.1007/s10701-009-9289-6

[19] D. J. Cirilo-Lombardo, Found. Phys. 39 (2009) 373–396.
https://doi.org/10.1007/s10701-009-9289-6

[20] J. R. Klauder and B. S. Skagerstam, *Coherent States: Applications in Physics and Mathematical Physics* (World Scientific, Singapore, 1985)

[21] John R. Klauder and E. C. G. Sudarshan, *Fundamentals of Quantum Optics*, Dover Publications (2006)

[22] A. M. Perelomov, *Generalized Coherent States and Their Applications*, Springer Berlin, Heidelberg (1986).

[23] K. Kowalski et al J. Phys. A: Math. Gen. 29 4149 (1996).
https://doi.org/10.1088/0305-4470/29/14/034

[24] K. Kowalski and J. Rembielinski, Phys. Rev. A 75 052102, (2007).

https://doi.org/10.1103/physreva.75.052102

[25] D. J. Cirilo-Lombardo, J. Phys. A: Math. Theor. 45, 244026 (2012).
https://doi.org/10.1088/1751-8113/45/24/244026

[26] A. M. Perelomov, *Coherent states for arbitrary Lie group*, Commun. Math. Phys. 26 (1972) 222–236].
https://doi.org/10.1007/bf01645091