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New Bounds for the Mass of Warm Dark Matter Particles Using Results from Fermionic King Model

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keV Warm Dark Matter in Agreement with Observations in Tribute to Hector de Vega
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Outline

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Motivations of the model and its characteristic lengths and masses

Short and long-range singularities

Isothermal sphere

Conventional (Boltzmann-Gibbs) thermo-statistical description for a self-gravitating non-relativistic system of point particles:

$$H = \sum_i \frac{1}{2m}\mathbf{p}_i^2 - \sum_{i<j} \frac{Gm^2}{|\mathbf{r}_i - \mathbf{r}_j|}, \tag{1}$$

leads to the distribution profile known of **isothermal sphere**:

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}. \tag{2}$$

The velocity dispersion is related to the temperature as $\sigma_v^2 = kT/m$.

This solution undergoes two divergences:

- The **short range singularity** (divergence of density for small distances $r \to 0$) associated with the phenomenon of **gravitational collapse**.
- The **long-range singularity** (the divergence of the total mass of the profile (2):

$$M = \int_0^{+\infty} \rho(r) 4\pi r^2 dr = +\infty. \tag{3}$$

Gravitation is unable to confine the constituents of any finite astrophysical system (escape of constituents is an unavoidable process).

Arguments to deal with these divergences

- The nature of **post-collapse stages** crucially depends on additional physics: *constituents with finite radius* (e.g.: hard or soft core particles models) or even the consideration of *quantum properties of micro-particles* (e.g: degenerate fermions at zero temperature, like models of white dwarfs and neutron stars).
- Evaporation imposes a **truncation of velocity spectrum** which is describe by a cutoff escape energy $\varepsilon_c < 0$:

$$f\left(\mathbf{r}, \mathbf{p}\right) = 0 \tag{4}$$

for energies $\varepsilon\left(\mathbf{r}, \mathbf{p}\right) > \varepsilon_c$, which provokes that density profile vanishes outside the called called as *tidal radius* R:

$$\rho(r) = 0 \text{ for } r > R. \tag{5}$$

This parameter is defined by the gravitational influence of other neighboring systems:

$$R \sim \left(\frac{M}{M_O}\right)^{1/3} R_O. \tag{6}$$

Fermionic-King model

Fermionic-King model:

$$f(\mathbf{r},\mathbf{p}|\beta,\varepsilon_c)=\frac{e^{\beta[\varepsilon_c-\varepsilon(\mathbf{r},\mathbf{p})]}-1}{\alpha+e^{\beta[\varepsilon_c-\varepsilon(\mathbf{r},\mathbf{p})]}}H[\varepsilon_c-\varepsilon(\mathbf{r},\mathbf{p})] \quad (7)$$

was proposed by Ruffini and Stella (1983) in the context of dark matter halos problems and theoretically re-derived by Chavanis (1998).

- $H(x)$ is Heaviside step function,
- $\varepsilon(\mathbf{r},\mathbf{p})=\mathbf{p}^2/2m+m\phi(\mathbf{r})$ denotes the individual mechanical energy for a particle with mass m and momentum $\mathbf{p}$ that is located at the position $\mathbf{r}$,
- $\beta=1/kT$ represents the inverse temperature parameter, $\varepsilon_c=m\phi_s$ denotes the energy threshold for the escape of particles, with ϕ_s being the surface potential at the tidal radius:

$$\phi_s=-\frac{GM}{R}, \quad (8)$$

- α is a dimensionless positive parameter associated with normalization of the one-body distribution and its degeneration.

Limit cases of this model

The ansatz (7) provides a suitable interpolation between the known **Fermi-Dirac distribution**:

$$f_{FD}\left(\mathbf{r},\mathbf{p}|\beta\right)=\frac{1}{e^{\beta\left[\varepsilon(\mathbf{r},\mathbf{p})-\varepsilon_F\right]}+1} \tag{9}$$

in the limit of low energies and the quasi-stationary one-body distribution associated with **King model**:

$$f_{K}\left(\mathbf{r},\mathbf{p}|\beta,\varepsilon_c\right)=\frac{1}{\alpha}\left[e^{\beta[\varepsilon_c-\varepsilon(\mathbf{r},\mathbf{p})]}-1\right]H\left[\varepsilon_c-\varepsilon\left(\mathbf{r},\mathbf{p}\right)\right] \tag{10}$$

in the classical non-degenerate limit. The Fermi energy ε_F and the escape energy enable us to rewrite the normalization constant as $\alpha\equiv e^{\beta\left(\varepsilon_c-\varepsilon_F\right)}\equiv\exp\left(\mu\right)$, where μ is referred to as degeneration parameter.

Characteristic lengths

Fermi-King model is non-relativistic and exhibits **two characteristic lengths**:

- **The tidal radius** R (classical regime) the system linear size at the high energy limit.
- **The Fermi radius** (the linear size of the degenerate self-gravitating system at zero temperature) R_F

$$R_F^3 \sim \frac{1}{M}\frac{\hbar^6}{G^3 m^8}, \tag{11}$$

which characterizes low energy limit.

The relativistic extension of Fermi-King model will exhibit **a third characteristic length!**:

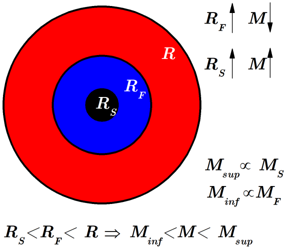
- **The Schwarzschild radius** $R_S = 2GM/c^2$ (relativistic regime), which appears when the total mass M is highly enough.

Characteristic masses

- The system stability against evaporation requires that the Fermi radius R_F should be lower than the tidal radius R, which implies the existence of a lower bound for the total mass M

$$R_F < R \rightarrow M > M_{inf} \propto M_F \sim \frac{1}{R^3} \frac{\hbar^6}{G^3 m^8}, \qquad (12)$$

where M_F is hereinafter denotes as the **Fermi mass**. The system whose total mass M is less than the lower bound M_{inf} is unable to confine particles and suffers from a complete evaporation disruption.



- The system stability against gravitational collapse in the relativistic regime requires that the Fermi radius R_F should be greater than the Schwarzschild radius R_S, which implies the existence of an upper bound for the total mass M

$$R_F > R_S \rightarrow M < M_{sup} \propto M_S \sim \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m^2}, \qquad (13)$$

where M_S can be denoted as the **Schwarzschild mass** (e.g.: Chandrasekhar and Tolman-Oppenheimer-Volkoff stability limits of stars). The system whose total mass M is greater than the upper bound M_{sup} is unable to avoid its gravitational collapse.

The numerical problem

The dimensionless potential $\Phi(\xi) = \beta m \left[\phi_s - \phi(\mathbf{r})\right]$ obeys the Poisson differential equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\Phi(\xi)}{d\xi}\right] = -4\pi F\left[\Phi(\xi), \mu, \frac{3}{2}\right], \tag{14}$$

corresponding to spherical solutions. Here, the function $F\left(\Phi, \mu, \nu\right)$:

$$F\left(\Phi, \mu, \nu\right) \equiv \frac{1}{\Gamma\left(\nu\right)}\int_0^{\Phi}\frac{e^{\Phi - x} - 1}{1 + e^{\Phi - x - \mu}}x^{\nu - 1}dx \tag{15}$$

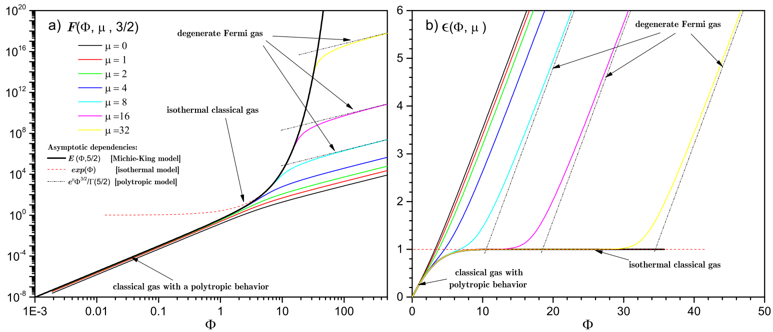
is referred to as the *Fermi-King integral*. The direct variables to numerically solve this non-linear problem is the central dimensionless potential $\Phi_0 = \Phi(0)$ and the degeneration parameter μ.

Systematic study of thermodynamics

- **Previous studies**: Thermodynamics of this model at constant (U, μ, M) [Chavanis at al (2015)].
- **This work**: Thermodynamics at constant $(U, R, M) \equiv (U, V, M)$. Here, $V = 4\pi R^3/3$ is the system volume enclosed inside the tidal radius R, while U and M are the total energy and the total mass.

Thermodynamics at constant total mass and tidal radius

Behavior of Fermi-King integrals



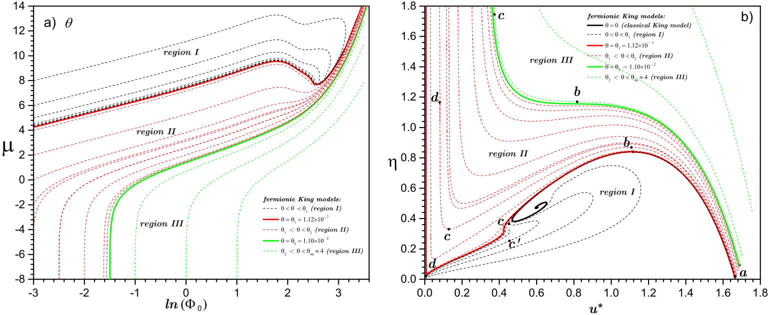
The function $\epsilon\left(\mathbf{r}\right)$ defined by the ratio between the kinetic energy per particle and the temperature:

$$\epsilon\left(\mathbf{r}\right)=m\beta\frac{2}{3}\frac{\upsilon(\mathbf{r})}{\rho(\mathbf{r})}\equiv\frac{1}{3}\beta m\left\langle v^{2}(\mathbf{r})\right\rangle\equiv\frac{F\left[\Phi(\mathbf{r}),\mu,5/2\right]}{F\left[\Phi(\mathbf{r}),\mu,3/2\right]}\qquad(16)$$

is local indicator concerning to the internal thermodynamic conditions at the position $\mathbf{r}$

Predicted distributions: King profiles (polytropic + isothermal cores and polytropic haloes) + Fermi-King profiles [degenerate (polytropic) + degenerate cores and polytropic haloes + degenerate cores, isothermal envelop and polytropic haloes].

Thermodynamics at constant total mass and tidal radius



The characteristic lengths (R, R_F) can be employed to introduce the mass ratio parameter θ:

$$\theta = \left(\frac{R_F}{R}\right)^3 \equiv \frac{M_F}{M}, \tag{17}$$

which is employed to perform calculations of thermodynamics variables: degeneration parameter μ, the dimensionless inverse energy $u^* = -GM^2/RU$ and the inverse temperature $\eta = \beta GMm/R$.

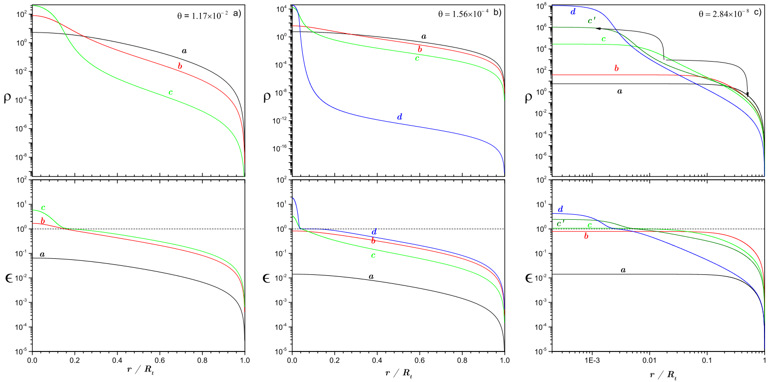
Thermodynamics at constant total mass and tidal radius

Critical masses associated with evaporation and quantum degeneration:

- **The region I**: the interval $0 \leq \theta < \theta_1 \simeq 1.12 \times 10^{-7}$ (black curves). The gravitational collapse of fermionic King model represents a discontinuous microcanonical phase transition, and its thermodynamics exhibits a branch with negative heat capacities. In terms of the total mass M, this region corresponds to situations with high total masses, the interval $M_1 < M < +\infty$, where $M_1 = M_F/\theta_1 \simeq 8.9 \times 10^6 M_F$.
- **The region II**: the interval $\theta_1 \leq \theta < \theta_2 \simeq 1.10 \times 10^{-2}$ (red curves). The gravitational collapse of fermionic King model turns a continuous microcanonical phase transition, and its thermodynamics exhibits a branch with negative heat capacities. In terms of the total mass M, this region corresponds to situations with intermediate total masses, the interval $M_2 < M \leq M_1$, where $M_2 = M_F/\theta_2 \simeq 90.9 M_F$.
- **The region III**: the interval $\theta_2 \leq \theta < \theta_m \simeq 4.0$ (green curves). The gravitational collapse of fermionic King model is a continuous microcanonical phase transition, and its thermodynamics does not exhibit negative heat capacities. In terms of the total mass M, this region corresponds to situations with low total mass, the interval $M_3 < M \leq M_2$, where $M_3 = M_F/\theta_m \simeq ¼ M_F$.

There is no solutions for $\theta > 4.0 \Leftrightarrow M < ¼ M_F$! The gravitation field of a fermion self-gravitating system whose total mass M is lower than $¼ M_F$ is unable to avoid its evaporation disruption.

Distribution profiles with degenerate cores for low energies



Application to dark matter haloes

Quantum degeneration and core-cusp problem

- Degenerate quantum properties of particles can avoid the **core-cusp problem** predicted by distributions like **NFW-profile**:

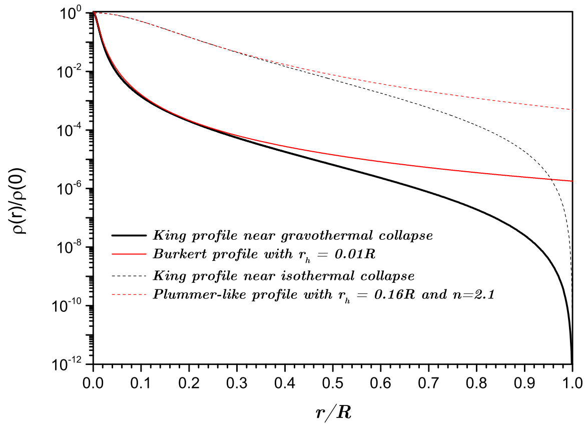
$$\rho_{NFW}(r) = \frac{\rho_0 r_s^3}{r\,(r_s + r)^2}. \tag{18}$$

- Considering the empirical rotation curves of large galaxies using the **Burkert profile**:

$$\rho_B(r) = \frac{\rho_0 r_h^3}{(r_h + r)\left(r_h^2 + r^2\right)}, \tag{19}$$

de Vega and co-workers (2013) have shown a great agreement of the Thomas-Fermi model in the classical regime within halos cores for distances $r < 2r_h$. **However, their analysis disregards the incidence of evaporation.**

Evaporation effects: Halo radius r_h *versus* tidal radius R



A comparison between a density profile near gravothermal collapse corresponding to King model and the Burkert profile (19) with fitting radius $r_h = 0.01R$. Despite the existing of small discrepancies in the innermost regions, the fit is reasonably good for distances $r < 25r_h = 0.25R$.

Bounds for WDM particles masses using fermionic King model

- Dwarf galaxies are rich in dark matter content $\Rightarrow$ fermionic King model for identical particles with mass m.
- Considering the multiplicity $g = 2s + 1 = 2$ and the rough estimation $R \simeq 100 r_h$ for the tidal radius R from the halos radius of Burkert profile r_h, the mass-ratio (17) can be expressed as follows:

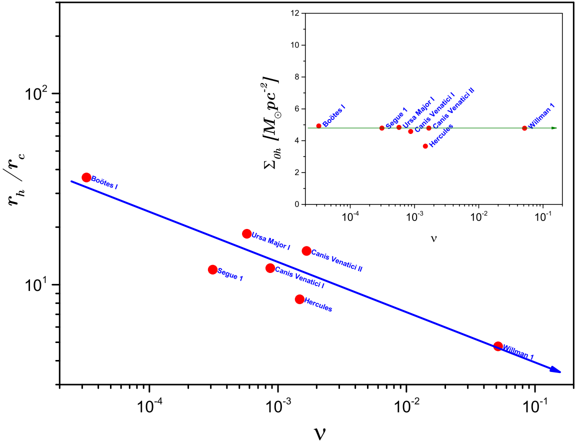
$$\theta = \frac{M_F}{M} = 17 \left(\frac{10^6 M_\odot}{M} \right) \left(\frac{pc}{r_h} \right)^3 \left(\frac{2\mathrm{keV}}{m} \right)^8 . \tag{20}$$

- Assuming that **Willman 1** (the lightest known dwarfs galaxy) belongs to the region III in the thermodynamics of fermionic King model, $M_3 < M < M_2$, the mass of WDM particle should belong to the interval:

$$m_{\min} = 1.2 \text{ keV} \leq m \leq m_{\max} = 2.6 \text{ keV}. \tag{21}$$

Galaxy	r_h [pc]	ρ_0 $[M_\odot\, pc^{-3}]$	M_h $[10^6 M_\odot]$	θ_{min} $[m_{max}]$	θ_h	θ_{max} $[m_{min}]$
Willman 1	19	6.3	0.029	1.1×10^{-2}	0.085	4.0
Segue 1	48	2.5	1.93	1.0×10^{-5}	8.0×10^{-5}	3.7×10^{-3}
Coma-Berenices	123	2.09	0.14	8.4×10^{-6}	6.5×10^{-5}	3.0×10^{-3}
Leo T	170	0.79	12.9	3.5×10^{-8}	2.7×10^{-7}	1.2×10^{-5}
Canis Venatici II	245	0.49	4.8	3.1×10^{-8}	2.4×10^{-7}	1.1×10^{-5}
Draco	305	0.5	26.5	2.9×10^{-9}	2.3×10^{-8}	1.0×10^{-6}
Leo II	320	0.34	36.6	1.8×10^{-9}	1.4×10^{-8}	6.6×10^{-7}
Hercules	387	0.1	25.1	1.5×10^{-9}	1.2×10^{-8}	5.5×10^{-7}
Boötes I	362	0.38	43.2	1.1×10^{-9}	8.3×10^{-9}	3.9×10^{-7}
Carina	428	0.15	32.2	8.7×10^{-10}	6.7×10^{-9}	3.2×10^{-7}
Ursa Major I	504	0.25	33.2	5.1×10^{-10}	4.0×10^{-9}	1.9×10^{-7}
Sculptor	480	0.25	78.8	2.5×10^{-10}	2.0×10^{-9}	9.1×10^{-8}
Leo IV	400	0.19	200	1.7×10^{-10}	1.3×10^{-9}	6.2×10^{-8}
Leo I	518	0.22	96	1.6×10^{-10}	1.3×10^{-9}	6.0×10^{-8}
Ursa Minor	750	0.16	193	2.7×10^{-11}	2.1×10^{-10}	9.8×10^{-9}
NGC 185	450	4.09	975	2.5×10^{-11}	1.9×10^{-10}	9.0×10^{-9}
Sextans	1290	0.02	116	8.8×10^{-12}	6.8×10^{-11}	3.2×10^{-9}
Canis Venatici I	1220	0.08	344	3.5×10^{-12}	2.7×10^{-11}	1.3×10^{-9}
Fornax	1730	0.053	1750	2.4×10^{-13}	1.9×10^{-12}	8.8×10^{-11}
NGC 855	1063	2.64	8340	2.2×10^{-13}	1.7×10^{-12}	7.9×10^{-11}
NGC 4478	1890	3.7	6.55×10^{4}	4.9×10^{-15}	3.8×10^{-14}	1.8×10^{-12}
Small Spiral	5100	0.029	6900	2.4×10^{-15}	1.8×10^{-14}	8.7×10^{-13}
NGC 3853	5220	0.77	2.87×10^{5}	5.4×10^{-17}	4.2×10^{-16}	1.9×10^{-14}
NGC 731	6160	0.47	2.87×10^{5}	3.3×10^{-17}	2.5×10^{-16}	1.2×10^{-14}
NGC 499	7700	0.91	1.09×10^{6}	4.4×10^{-18}	3.4×10^{-17}	1.6×10^{-15}
Medium Spiral	1.9×10^{4}	0.0076	1.01×10^{5}	3.2×10^{-18}	2.4×10^{-17}	1.1×10^{-15}
Large Spiral	5.9×10^{4}	2.3×10^{-3}	1.0×10^{6}	1.1×10^{-20}	8.3×10^{-20}	3.9×10^{-18}

Baryonic/DM mass ratio $\nu = M_B/M_{DM}$ for some dSph galaxies



The relative large baryonic content of Willman 1 is consistent with a significant mass-loss of its DM halo.

Finals considerations

Conclusions and open questions

Fermionic King model predicts:

- A **post-collapse branch of stability** for low energies (with profiles with degenerate cores).
- An **evaporation disruption** if its total mass $M < \frac{1}{4} M_F$.
- The total mass M drives the character continuous or discontinuous of **microcanonical phase transition associated with gravothermal collapse** and the existence or nonexistence of states with negative heat capacities.

Application to dark matter haloes:

- Structural parameters of dwarfs galaxies like Willman 1 are consistent with **WDM particles with masses in the keV scale**.
- Most of galaxies must develop a **degenerate core of WDM particles** via a sudden (violent) gravothermal collapse.

Open question: Relativistic extension of fermionic King model

The quantum-relativistic conditions governing the interior of galactic cores of large galaxies are somehow correlated to the masses of these stellar systems. Considering the Schwarzschild mass (13) for a WDM particle with mass $m = 2$ keV, one obtains:

$$M_c \sim 6.4 \times 10^{12} M_\odot . \qquad (22)$$

Remarkably, this mass scale is **comparable to the typical masses of large galaxies**. The present argument suggests the masses of the large galaxies are related to quantum-relativistic processes involving WDM particles (e.g.: the formation of **supermassive black holes**). The characteristic mass (22) could explain the observed properties of large galaxies if DM halos are composed of WDM particles with masses in the keV scale.

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Update concerning extension of fermionic King model to GR

The extension of fermionic King model for general relativity was recently proposed in

[10] Argüelles et al, MNRAS **502**, 4227 (2021)

presenting a thermodynamic analysis *at constant degeneration parameter* $\mu = (\varepsilon_c - \varepsilon_F)/kT$ *and total mass* M. Notice that the degeneration parameter μ is closely related to *chemical potential* ($w = \varepsilon_c - \varepsilon_F$ is the amount of energy required to remove a particle from the system). The degeneration parameter μ appears here as **the variable thermodynamic conjugated to the number of particles** N (or total mass $M = Nm$) with respect the entropy. A statistical ensemble that guarantees the constancy of (μ, M) is quite analogous to a statistical ensemble that guarantees the constancy of the temperature and energy (T, U). From viewpoint of statistical mechanics, however, *there not exist statistical ensembles that guarantee the simultaneous constancy of thermodynamic conjugated variables!* To avoid this thermo-statistical inconsistence, the thermodynamics at constant tidal radius R and total mass M is still required in order to obtain relativistic mass bound M_S that completes the critical masses (M_1, M_2, M_3) obtained in the work:

[11] L. Velazquez, *Universe* **7**, 308 (2021)

Thanks

Acknowledgments

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