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Model-independent analysis of dark matter points to a particle mass at the keV scale

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ABSTRACT

We present a *model-independent* analysis of dark matter (DM) decoupling both ultrarelativistically (UR) and non-relativistically (NR) based on the DM phase-space density $\mathcal{D} = \rho_{\rm DM}/\sigma^3_{\rm DM}$. We derive explicit formulae for the DM particle mass m and for the number of ultrarelativistic degrees of freedom $g_{\rm d}$ at decoupling. We find that for DM particles decoupling UR both at local thermal equilibrium (LTE) and out of LTE, m turns out to be in the *keV scale*. For example, for DM Majorana fermions decoupling at LTE the resulting mass is $m \simeq 0.85$ keV. For DM particles decoupling NR, $\sqrt{m T_{\rm d}}$ results in the keV scale ($T_{\rm d}$ is the decoupling temperature) and the value of m is consistent with the keV scale. In all cases, DM turns out to be *cold* DM (CDM). In addition, lower and upper bounds on the DM annihilation cross-section for NR decoupling are derived. We evaluate the free-streaming (Jeans) wavelength and Jeans mass: they are independent of the type of DM except for the DM self-gravity dynamics. The free-streaming wavelength today turns to be in the *kpc range*. These results are based on our theoretical analysis, on astronomical observations of dwarf spheroidal satellite galaxies in the Milky Way and on N-body numerical simulations. We analyse and discuss the results for $\mathcal{D}$ from analytic approximate formulae for both linear fluctuations and the (non-linear) spherical model and from N-body simulations results. In this way we obtain upper bounds for the DM particle mass, which are all below the 100-keV range.

Key words: galaxies: fundamental parameters – cosmology: theory – dark matter.

1 THE DARK MATTER PARTICLE MASS

Although the existence of dark matter (DM) was proposed 75 years ago (Zwicky 1933; Oort 1940, see van den Bergh 2001 for a history of the research on dark matter) its nature is as yet unknown. It must have been non-relativistic by the time of structure formation ($z < 30$) in order to reproduce the observed small structure at ∼2–3 kpc.

DM particles can decouple ultrarelativistically (UR) at $T_{\rm d} \gg m$ or non-relativistically (NR) at $T_{\rm d} \ll m$, where m is the mass of the DM particles and $T_{\rm d}$ the decoupling temperature. We consider in this paper particles that decouple at or out of local thermal equilibrium (LTE).

The DM distribution function $F_{\rm d}$ freezes out at decoupling. Therefore, for all times after decoupling $F_{\rm d}$ coincides with its expression at decoupling. $F_{\rm d}$ is a function of $T_{\rm d}, m$ and the comoving momentum of the DM particles $p_{\rm c}$.

Knowing the distribution function $F_{\rm d}(p_{\rm c})$, we can compute physical magnitudes as the DM velocity fluctuations and the DM energy density. For the relevant times t during structure formation, when the DM particles are non-relativistic, we have

$$\langle \boldsymbol{V}^2\rangle(t) = \left\langle \frac{\boldsymbol{p}^2_{\rm ph}}{m^2}\right\rangle(t) = \frac{\displaystyle\int \frac{{\rm d}^3 p_{\rm ph}}{(2\pi)^3}\, \frac{\boldsymbol{p}^2_{\rm ph}}{m^2}\, F_{\rm d}[a(t)\, p_{\rm ph}]}{\displaystyle\int \frac{{\rm d}^3 p_{\rm ph}}{(2\pi)^3}\, F_{\rm d}[a(t)\, p_{\rm ph}]}, \tag{1}$$

where we use the physical momentum of the DM particles $p_{\rm ph}(t) \equiv p_{\rm c}/a(t)$ as the integration variable. The scale factor $a(t)$ is normalized as usual:

$$a(t) = \frac{1}{1 + z(t)}, \quad a(\text{today}) = 1; \tag{2}$$

that is, the physical momentum $p_{\rm ph}(t)$ coincides today with the comoving momentum $p_{\rm c}$ at zero redshift.

We can relate the covariant decoupling temperature, $T_{\rm d}$, the effective number of UR degrees of freedom at decoupling, $g_{\rm d}$, and the photon temperature today, T_γ, using entropy conservation (Kolb & Turner 1990; Börner 2003; Yao et al. 2006):

$$T_{\rm d} = \left(\frac{2}{g_{\rm d}}\right)^{1/3} T_\gamma, \quad \text{where} \quad T_\gamma = 0.2348 \text{ meV} \tag{3}$$

and 1 meV $= 10^{-3}$ eV.

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The DM energy density can be written as

$$\rho_{\rm DM}(t) = g \int \frac{{\rm d}^3 p_{\rm ph}}{(2\pi)^3} \sqrt{m^2 + p_{\rm ph}^2} F_{\rm d}[a(t) p_{\rm ph}], \tag{4}$$

where g is the number of internal degrees of freedom of the DM particle, typically $1 \le g \le 4$.

By the time the DM particles are non-relativistic, the energy density equation (4) has become

$$\rho_{\rm DM}(t) = \frac{m g}{2\pi^2} \frac{T_{\rm d}^3}{a^3(t)} I_2 \equiv m n(t), \tag{5}$$

where

$$I_2 \equiv \int_0^\infty y^2 F_{\rm d}(y)\, {\rm d}y,$$

$n(t)$ is the number of DM particles per unit volume, and we have used as integration variable

$$y \equiv \frac{p_{\rm ph}(t)}{T_{\rm d}(t)} = \frac{p_{\rm c}}{T_{\rm d}}. \tag{6}$$

From equation (5) at $t = 0$ and from the value observed today for $\rho_{\rm DM}$ (Komatsu et al. 2009; Yao et al. 2006),

$$\begin{aligned} \rho_{\rm DM} &= \Omega_{\rm DM} \rho_{\rm c} = 0.228 \rho_{\rm c}, \\ \rho_{\rm c} &= 3 M_{\rm Pl}^2 H_0^2 = (2.518\ {\rm meV})^4, \end{aligned} \tag{7}$$

and $M_{\rm Pl}^2 = 1/[8\pi G]$, we find the value of the DM mass particle:

$$m = \pi^2 \Omega_{\rm DM} \frac{\rho_{\rm c}}{T_\gamma^3} \frac{g_{\rm d}}{g I_2} = 6.986\ {\rm eV} \frac{g_{\rm d}}{g I_2}, \tag{8}$$

where $\rho_{\rm c}$ is the critical density.

Using y as the integration variable (equation 6), equation (1) for the velocity fluctuations yields

$$\langle \boldsymbol{V}^2 \rangle(t) = \left[\frac{T_{\rm d}}{m a(t)}\right]^2 \frac{I_4}{I_2}, \tag{9}$$

where

$$I_4 \equiv \int_0^\infty y^4 F_{\rm d}(y)\, {\rm d}y.$$

Expressing $T_{\rm d}$ in terms of the cosmic microwave background (CMB) temperature today according to equation (3) gives, for the one-dimensional velocity dispersion,

$$\sigma_{\rm DM}(z) = \sqrt{\frac{1}{3} \langle \boldsymbol{V}^2 \rangle(z)} = \frac{2^{1/3}}{\sqrt{3}} \frac{1+z}{g_{\rm d}^{1/3}} \frac{T_\gamma}{m} \sqrt{\frac{I_4}{I_2}} \tag{10}$$

$$= 0.05124 \frac{1+z}{g_{\rm d}^{1/3}} \frac{\rm keV}{m} \left[\frac{I_4}{I_2}\right]^{1/2} {\rm km\, s^{-1}}. \tag{11}$$

It is very useful to consider the invariant phase-space density under universe expansion (Boyanovsky, de Vega & Sanchez 2008a; Hogan & Dalcanton 2000; Dalcanton & Hogan 2001; Madsen 1990, 2001):

$$\mathcal{D}(t) \equiv \frac{n(t)}{\left\langle \boldsymbol{p}_{\rm ph}^2(t) \right\rangle^{3/2}} \overset{\rm non-rel}{=} \frac{1}{3\sqrt{3} m^4} \frac{\rho_{\rm DM}(t)}{\sigma_{\rm DM}^3(t)}, \tag{12}$$

where we consider the relevant times t during structure formation when the DM particles are non-relativistic. $\mathcal{D}(t)$ is a *constant* in the absence of self-gravity. In the non-relativistic regime, $\mathcal{D}(t)$ can decrease only by collisionless phase mixing or self-gravity dynamics (Lynden-Bell 1967; Tremaine, Henon & Lynden-Bell 1986).

We derive a useful expression for the phase-space density $\mathcal{D}$ from equations (5), (10) and (12), with the result

$$\mathcal{D} = \frac{g}{2\pi^2} \frac{I_2^{5/2}}{I_4^{3/2}}. \tag{13}$$

Observations of dwarf spheroidal satellite galaxies (dSphs) in the Milky Way yield for the phase-space density today (Wyse & Gilmore 2007; Gilmore et al. 2007):

$$\frac{\rho_{\rm s}}{\sigma_{\rm s}^3} \sim 5 \times 10^3 \frac{\rm keV\, cm^{-3}}{({\rm km\, s^{-1}})^3} = (0.18\ {\rm keV})^4. \tag{14}$$

The precision of these results is about a factor of 10.

After the radiation-dominated era, the phase-space density decreases by a factor that we denote by Z:

$$\mathcal{D}(0) = \frac{1}{Z} \mathcal{D}(z \sim 3200). \tag{15}$$

Recall that $\mathcal{D}(z) = \rho_{\rm DM}/(3\sqrt{3} m^4 \sigma_{\rm DM}^3)$ (according to equation 12) is independent of z for $z \gtrsim 3200$, as density fluctuations were $\lesssim 10^{-3}$ before the matter-dominated era (Dodelson 2003).

The range of values of Z (which is necessarily $Z > 1$) is analysed in detail in Section 2.3 below.

We can express the phase-space density today from equations (12) and (14) as

$$\mathcal{D}(0) = \frac{1}{3\sqrt{3} m^4} \frac{\rho_{\rm s}}{\sigma_{\rm s}^3}. \tag{16}$$

Therefore, equations (12), (15) and (16) yield

$$\frac{\rho_{\rm s}}{\sigma_{\rm s}^3} = \frac{1}{Z} \frac{\rho_{\rm DM}}{\sigma_{\rm DM}^3}(z \sim 3200), \tag{17}$$

where $\rho_{\rm DM}/\sigma_{\rm DM}^3(z \sim 3200)$ follows from equations (12) and (13):

$$\frac{\rho_{\rm DM}}{\sigma_{\rm DM}^3}(z \sim 3200) = \frac{3\sqrt{3} m^4}{2\pi^2} g \frac{I_2^{5/2}}{I_4^{3/2}}. \tag{18}$$

We can express m from equations (14)–(18) in terms of $\mathcal{D}$ and observable quantities as

$$m^4 = \frac{Z}{3\sqrt{3}} \frac{\rho_{\rm s}}{\mathcal{D} \sigma_{\rm s}^3} = \frac{2\pi^2}{3\sqrt{3}} \frac{Z}{g} \frac{\rho_{\rm s}}{\sigma_{\rm s}^3} \frac{I_4^{3/2}}{I_2^{5/2}}, \tag{19}$$

$$m = 0.2504 \left(\frac{Z}{g}\right)^{1/4} \frac{I_4^{3/8}}{I_2^{5/8}}\ {\rm keV}. \tag{20}$$

Combining this with equation (8) for m we obtain the number of ultrarelativistic degrees of freedom at decoupling as

$$\begin{aligned} g_{\rm d} &= \frac{2^{1/4}}{3^{3/8} \pi^{3/2}} \frac{g^{3/4}}{\Omega_{\rm DM}} \frac{T_\gamma^3}{\rho_{\rm c}} \left(\frac{Z \rho_{\rm s}}{\sigma_{\rm s}^3}\right)^{1/4} [I_2 I_4]^{3/8} \\ &= 35.96\, Z^{1/4} g^{3/4} [I_2 I_4]^{3/8}. \end{aligned} \tag{21}$$

If we assume that DM today is a self-gravitating gas in thermal equilibrium described by an isothermal sphere solution of the Lane–Emden equation, the relevant quantity characterizing the dynamics is the dimensionless variable

$$\eta = \frac{G m^2 N}{L T} = \frac{2}{3} G L^2 \frac{\rho_{\rm s}}{\sigma_{\rm s}^2} \tag{22}$$

(de Vega & Sánchez 2002a,b; Destri & de Vega 2007), which is bound to be $\eta \lesssim 1.6$ to prevent the gravitational collapse of the gas (de Vega & Sánchez 2002a,b; Destri & de Vega 2007). Here, $V = L^3$ denotes the volume occupied by the gas, N the number of particles, G Newton's constant, and $T = (3/2) m\sigma^2$ is the gas temperature. [The length L is similar to the so-called King radius (Binney & Tremaine 1987). Note, however, that the King radius follows from the singular isothermal sphere solution whereas L is the characteristic size of a stable isothermal sphere solution (de Vega & Sánchez 2002a,b; Destri & de Vega 2007).]

The compilation of recent photometric and kinematic data from 10 Milky Way dSphs satellites (Wyse & Gilmore 2007; Gilmore et al. 2007) yields values for the one-dimensional velocity dispersion σ_s and the radius L in the ranges

$$0.5\,\text{kpc} \le L \le 1.8\,\text{kpc}, \quad 6.6\,\text{km}\,\text{s}^{-1} \le \sigma_s \le 11.1\,\text{km}\,\text{s}^{-1}. \tag{23}$$

Combining equations (12), (17) and (22) yields an explicit expression for the DM particle mass:

$$m^4 \sim \frac{1}{2\sqrt{3}G}\,\eta\,\frac{Z}{L^2\mathcal{D}\sigma_s} = \frac{4\pi}{\sqrt{3}}M_{\rm Pl}^2\eta\,\frac{Z}{L^2\mathcal{D}\sigma_s}$$
$$= 0.5279\times10^{-4}\,\frac{\eta Z}{\mathcal{D}}\,\frac{10\,\text{km}\,\text{s}^{-1}}{\sigma_s}\,\left(\frac{\text{kpc}}{L}\right)^2 (\text{keV})^4. \tag{24}$$

This formula provides an expression for the DM particle mass independent of equation (19). We shall see below that equations (19) and (24) yield similar results.

In the subsequent sections we investigate cases in which DM particles decouple UR or NR both at LTE and out of LTE. We compute m and g_d explicitly in the various cases according to the general formulae (20), (21) and (24).

1.1 Jeans (free-streaming) wavelength and Jeans mass

It is important to evaluate the Jeans length and Jeans mass in the present context (Börner 2003; Gilbert 1968; Bond & Szalay 1983). The Jeans length is analogous to the free-streaming wavelength. The free-streaming wavevector is the largest wavevector exhibiting gravitational instability and characterizes the scale of suppression of the DM transfer function $T(k)$ (Boyanovsky, de Vega & Sanchez 2008b).

The physical free-streaming wavelength can be expressed as (Börner 2003; Boyanovsky et al. 2008b)

$$\lambda_{\rm fs}(t) = \lambda_{\rm J}(t) = \frac{2\pi}{k_{\rm fs}(t)}, \tag{25}$$

where $k_{\rm fs}(t) = k_{\rm J}(t)$ is the physical free-streaming wavenumber given by

$$k_{\rm fs}^2(t) = \frac{4\pi G\rho_{\rm DM}(t)}{\langle \boldsymbol{V}^2\rangle(t)} = \frac{3}{2}[1+z(t)]\,\frac{H_0^2\Omega_{\rm DM}}{\langle \boldsymbol{V}^2\rangle(0)}, \tag{26}$$

where we used that $\rho_{\rm DM}(t) = \rho_{\rm DM}(0)\,(1+z)^3$ and equation (7).

We obtain the primordial DM dispersion velocity $\sigma_{\rm DM}$ from equations (5), (7) and (17):

$$\sqrt{\frac{1}{3}\,\langle \boldsymbol{V}^2\rangle(0)} = \sigma_{\rm DM} = \left(3M_{\rm Pl}^2H_0^2\Omega_{\rm DM}\,\frac{1}{Z}\,\frac{\sigma_s^3}{\rho_s}\right)^{1/3}. \tag{27}$$

This expression is valid for any kind of DM particle. Inserting equation (27) into equation (26) yields the following expression for the physical free-streaming wavelength:

$$\lambda_{\rm fs}(z) = \frac{2\sqrt{2}\pi}{\Omega_{\rm DM}^{1/6}}\left(\frac{3M_{\rm Pl}^2}{H_0}\right)^{1/3}\left(\frac{\sigma_s^3}{Z\rho_s}\right)^{1/3}\frac{1}{\sqrt{1+z}}$$
$$= \frac{16.3}{Z^{1/3}}\,\frac{1}{\sqrt{1+z}}\,\text{kpc}, \tag{28}$$

where we used $1\,\text{keV} = 1.563738\times10^{29}\,(\text{kpc})^{-1}$.

Note that $\lambda_{\rm fs}$ and therefore $\lambda_{\rm J}$ are independent of the nature of the DM particle except for the factor Z.

The approximated analytic evaluations in Section 2 together with the results of N-body simulations (Peirani, Durier & de Freitas Pacheco 2006; Hoffman et al. 2007; Lapi & Cavaliere 2009; Romano-Diaz et al. 2006, 2007; Vass et al. 2009) indicate that, for dSphs, Z is in the range

$$1 < Z < 10000.$$

Therefore, $1 < Z^{1/3} < 21.5$ and the free-streaming wavelength is in the range

$$0.757\,\frac{1}{\sqrt{1+z}}\,\text{kpc} < \lambda_{\rm fs}(z) < 16.3\,\frac{1}{\sqrt{1+z}}\,\text{kpc}.$$

These values at $z = 0$ are consistent with the N-body simulations reported in Gao & Theuns (2007) and are of the order of the small DM structures observed today (Wyse & Gilmore 2007; Gilmore et al. 2007).

The Jeans mass is given by

$$M_{\rm J}(t) = \frac{4}{3}\pi\lambda_{\rm J}^3(t)\,\rho_{\rm DM}(t), \tag{29}$$

and provides the smallest unstable mass by gravitational collapse (Kolb & Turner 1990; Börner 2003). Inserting equation (5) for the DM density and equation (28) for $\lambda_{\rm J}(t) = \lambda_{\rm fs}(t)$ yields

$$M_{\rm J}(z) = 192\sqrt{2}\pi^4\,\sqrt{\Omega_{\rm DM}}M_{\rm Pl}^4H_0\,\frac{\sigma_s^3}{Z\rho_s}\,(1+z)^{3/2}$$
$$= \frac{0.4464}{Z}\,10^7\,\text{M}_\odot\,(1+z)^{3/2}. \tag{30}$$

Taking into account the range of Z-values yields

$$0.4464\times10^3 M_\odot < M_{\rm J}(z)\,(1+z)^{-3/2} < 0.4464\times10^7\,\text{M}_\odot.$$

This gives masses of the order of galactic masses $\sim10^{11}\,\text{M}_\odot$ by the beginning of the matter-dominated (MD) era $z\sim3200$. In addition, the comoving free-streaming wavelength at $z\sim3200$,

$$3200\times\lambda_{\rm fs}(z\sim3200)\sim100\,\text{kpc},$$

turns out to be of the order of the size of galaxies today.

2 THE PHASE-SPACE DENSITY $\mathcal{D}$ FROM ANALYTIC APPROXIMATION METHODS AND FROM N-BODY SIMULATIONS

We now analytically derive formulae for the reduction factor Z defined by equation (15) in the linear approximation and in the spherical model. The results obtained (see Table 1) are in fact *upper bounds* for Z. We analyse the results for $\mathcal{D}$ from N-body simulations.

2.1 Linear perturbations

The simplest calculation of $\mathcal{D}$ follows by considering linear perturbations around the homogeneous distribution $\rho_{\rm DM}(z)$ as

$$\rho = \rho_{\rm DM}(z)[1+\delta(z,k)]. \tag{31}$$

We have $\rho_{\rm DM}(z) = \rho_{\rm DM}(0)\,(1+z)^3$, and in a MD universe

$$\delta(z,k)\sim\delta_{\rm i}\,\frac{1+z}{1+z_{\rm i}}. \tag{32}$$

The peculiar velocity in the MD universe behaves as (Dodelson 2003)

$$v\sim aH\delta\sim1/\sqrt{1+z}. \tag{33}$$

We can thus relate the phase-space density $\mathcal{D}(z)\sim\rho/v^3$ at redshift z (equation 12) to the phase-space density at redshift $z_{\rm i}$ as

$$\mathcal{D}(z)\sim\mathcal{D}(z_{\rm i})\,\left(\frac{1+z}{1+z_{\rm i}}\right)^{9/2}. \tag{34}$$

Because the linear approximation is valid for $|\delta|^2\ll1$, we find from equation (32) that equation (34) applies in the redshift range $(z_{\rm i}, z)$, where

$$1+z\gg(1+z_{\rm i})\,\delta_{\rm i}. \tag{35}$$

Table 1. Upper bounds for the Z factor (defined by equation 15) and for the mass of the DM particle obtained for two different approximation methods. Note that only the spherical model takes into account non-linear self-gravity effects. The mass m depends weakly on Z through the power 1/4. In both cases m is in the keV range.

Approximation used	Upper limit on Z	Upper limit on $m \simeq 0.5\, Z^{1/4}$ keV
Linear fluctuations	$\sim 1.3 \times 10^{11}$	96 keV
Spherical model	$\sim 1.29 \times \delta_i^{-3/2} \simeq 4.1 \times 10^4$	7.1 keV

We can apply equation (34) to relate $\mathcal{D}(z)$ at equilibration ($z = z_i \simeq 3200$) and $\mathcal{D}(z)$ at the beginning of structure formation, $z \sim 30$, as $\delta_i \sim 10^{-3}$ is a scale average of the density fluctuations at the end of the radiation-dominated (RD) era (Dodelson 2003) and equation (35) is satisfied. We thus obtain from equation (34) in the linear approximation that

$$\frac{\mathcal{D}(z \simeq 3200)}{\mathcal{D}(z \simeq 30)} \sim 1.3 \times 10^9. \tag{36}$$

Note that equation (35) does not hold for $z \sim 0$. Therefore, in order to evaluate Z, we should combine the linear approximation result (equation 36) for $30 \lesssim z \lesssim 3200$ with the results of N-body simulations for $0 \lesssim z \lesssim 30$. This is done in Section 2.4 to obtain upper bounds for Z.

2.2 The spherical model

Let us now consider the spherical model in which particles move only in the radial direction but in which the non-linear evolution is exactly solved (Fillmore & Goldrich 1984a,b; Bertschinger 1985a,b; Peebles 1993; Padmanabhan 2000). The proper radius of the spherical shell obeys the equation

$$\ddot{R} = -\frac{GM}{R^2}, \tag{37}$$

where G is the gravitational constant and M is the (constant) mass enclosed by the shell. Equation (37) can be solved in close form with the solution (Bertschinger 1985a,b; Peebles 1993)

$$\begin{aligned} t &= \frac{3t_i}{4\delta_i^{3/2}}\,(\theta - \sin\theta)\,, \\ R &= \frac{R_i}{2\delta_i}\,(1-\cos\theta), \quad \frac{2R_i^3}{9t_i^2} = GM, \\ \dot{R} &= \frac{2R_i\sqrt{\delta_i}}{3t_i}\,\frac{\sin\theta}{1-\cos\theta}, \\ 1+z &= (1+z_i)\,\delta_i\,\left(\frac{4}{3}\right)^{2/3}\,\frac{1}{(\theta-\sin\theta)^{2/3}}, \\ \rho &= \rho_{\rm DM}(z)\,\frac{9}{2}\,\frac{(\theta-\sin\theta)^2}{(1-\cos\theta)^3}. \end{aligned} \tag{38}$$

Here, R_i and z_i are the radius and the redshift at the initial time t_i, and θ is an auxiliary time-dependent parameter.

Choosing the initial time by equilibration with $z_i \gg 1$ we have $\theta_i \ll 1$, and we find from equations (38) that

$$\theta_i = 2\sqrt{\delta_i}, \quad \dot{R}(t_i) = \frac{2R_i}{3t_i}, \quad \rho_i = \rho_{\rm DM}(z_i). \tag{39}$$

The spherical shell reaches its maximum radius of expansion $R_m = R_i/\delta_i$ at $\theta = \pi$ and then it turns around and collapses to a point at $\theta = 2\pi$. However, well before that, the approximation that matter only moves radially and that random particle velocities are small will break down. In fact, the DM relaxes to a virialized configuration in which the velocity and the virial radius follow from the virial theorem (Padmanabhan 2000)

$$v^2 = \frac{6GM}{5R_m}, \quad R_v = \frac{1}{2}R_m. \tag{40}$$

We can now compute the initial phase-space density (at z_i) and the phase-space density at virialization. At z_i we can use equations (12) and (39) to obtain

$$\mathcal{D}_i = \frac{\rho_{\rm DM}(0)}{3\sqrt{3}m^4}\,(1+z_i)^3\,\left(\frac{3t_i}{2R_i}\right)^3, \tag{41}$$

and at virialization for $\theta = 2\pi$ we obtain from equations (12) and (40) that

$$\mathcal{D}_v = \frac{\rho_{\rm DM}(0)}{3\sqrt{3}m^4}\,(1+z_i)^3\,\left(\frac{t_i}{R_i}\right)^3\,\frac{32}{9\pi^2}\,\left(\frac{15}{4}\,\delta_i\right)^{3/2}. \tag{42}$$

Therefore, the Z factor in the spherical model takes the value

$$Z = \frac{\mathcal{D}_i}{\mathcal{D}_v} = \frac{9\pi^2}{32}\,\left(\frac{3}{5\delta_i}\right)^{3/2} = \frac{1.29009}{\delta_i^{3/2}}.$$

Setting $\delta_i \sim 10^{-3}$ as a scale average of the density fluctuations at the end of the RD era (Dodelson 2003) yields

$$Z \sim 4.08 \times 10^4. \tag{43}$$

The spherical model approximates the evolution as a purely radial expansion followed by a radial collapse. Because no transverse motion is allowed, nor mergers, the spherical model result for Z (equation 43) is actually an upper bound on Z.

2.3 The phase-space density $\mathcal{D}$ from N-body simulations

The phase-space density $\mathcal{D}(z)$ is invariant under universe expansion except for the self-gravity dynamics that diminish $\mathcal{D}(z)$ in its evolution (Lynden-Bell 1967; Tremaine et al. 1986). Numerical simulations show that $\mathcal{D}(z)$ decreases sharply during phases of violent mergers which are followed by quiescent phases (Peirani et al. 2006; Hoffman et al. 2007; Lapi & Cavaliere 2009; Romano-Diaz et al. 2006, 2007; Vass et al. 2009). $\mathcal{D}(z)$ decreases during these violent phases by a factor of the order $\gtrsim 1$. (See fig. 3 in Peirani et al. 2006; fig. 1 in Hoffman et al. 2007; fig. 6 in Lapi & Cavaliere 2009 and fig. 5 in Vass et al. 2009.) These sharp decreases in $\mathcal{D}$ are in agreement with the linear approximation of Section 2.1, as shown below.

A succession of several violent phases occurs during the structure formation stage ($z \lesssim 30$). Their cumulated effect, together with the evolution of $\mathcal{D}$ for $3200 \gtrsim z \gtrsim 30$, produces a range of values for the Z factor that we can conservatively estimate on the basis of the results of N-body simulations (Peirani et al. 2006; Hoffman et al. 2007; Lapi & Cavaliere 2009; Romano-Diaz et al. 2006, 2007; Vass et al. 2009) and the approximation results of equations (36) and (43). This gives a range of values $1 < Z < 10\,000$ for dSphs. A more accurate analysis of N-body simulations should narrow this range for Z, which depends on the type and size of the galaxy considered.

The dSph observations, for which the best observational data on DM dominated galaxies are available, take into account mostly the cores of the structures, as dSphs have been stripped of their external haloes. Hence, the observed values of ρ_s/σ_s^3 may be higher than the space-averaged values represented on the right-hand sides of equations (15) and (17). Higher values for ρ_s/σ_s^3 correspond to lower values for Z.

The approximate formula (34) indicates a sharp decrease of the phase-space density with redshift. This sharp decrease is in qualitative agreement with simulations for the violent phases; see Peirani et al. (2006); Hoffman et al. (2007); Lapi & Cavaliere (2009); Romano-Diaz et al. (2006, 2007); Vass et al. (2009).

2.4 Synthetic discussion on the evaluation of Z and its upper bounds

The DM particle mass scale is set by the phase-space density for dSphs (equation 14). These galaxies are particularly dense and exhibit larger values for ρ_s/σ_s^3 than spiral galaxies. Because the primordial phase-space density is a universal quantity depending only on cosmological parameters, the Z factor must be galaxy-dependent – larger for spiral galaxies than for dSphs.

We want to stress that the values of the relevant quantities m and g_d are *weakly* affected by the uncertainty of Z through the factor $Z^{1/4}$ (see equations 20 and 21).

Equation (34) provides an extreme high estimate for the decrease of $\mathcal{D}$, and hence an extreme high estimate for Z. The N-body simulations show that the violent decrease of $\mathcal{D}$ is restricted to a factor of order one in each violent phase (Peirani et al. 2006; Hoffman et al. 2007; Lapi & Cavaliere 2009; Romano-Diaz et al. 2006, 2007; Vass et al. 2009).

In summary, the linear approximation suggests a reduction of $\mathcal{D}$ at each violent phase by a factor $\gtrsim 1$, where such an approximation is valid. Successive violent phases can reduce $\mathcal{D}$ by a factor of up to ~ 10 in the range $0 \lesssim z \lesssim 30$, as shown in the simulations (Peirani et al. 2006; Hoffman et al. 2007; Lapi & Cavaliere 2009; Romano-Diaz et al. 2006, 2007; Vass et al. 2009).

Combining the approximate decrease of $\mathcal{D}(z)$ given by equation (36) with an upper bound of a decrease by a factor ~ 100 for the interval $0 \lesssim z \lesssim 30$ yields in the linear approximation the upper bound

$$Z < 1.3 \times 10^{11}\ , \tag{44}$$

and we have equation (43) for Z in the spherical model. The fact that Z in the spherical model turns out to be several orders of magnitude below the Z value in the linear approximation arises from the fact that the spherical model includes non-linear effects and is therefore more reliable than the linear approximation.

The range $1 < Z < 10\,000$ for dSphs from N-body simulations corresponds to realistic initial conditions in the simulations.

The evolutions in the two approximations considered (see Table 1) are simple spatially isotropic expansions, followed by a collapse in the case of the spherical model. There is no possibility of non-radial motion nor of mergers in these approximations, in contrast to the case in N-body simulations. For such reasons, the Z values in Table 1 are upper bounds to the true values of Z in galaxies. The largest bound on Z yields DM particle masses below $\sim$100 keV. Moreover, the more reliable spherical model yields 7.1 keV as an upper bound for the DM particle mass.

In summary, with realistic initial conditions, $\mathcal{D}$ will not decrease more than a factor $\lesssim 10\,000$, and it is therefore fair to assume that $Z < 10\,000$ for dSphs.

3 DM PARTICLES DECOUPLING ULTRARELATIVISTICALLY

3.1 Decoupling at LTE

If DM particles of mass m decouple at a temperature $T_d \gg m$ their freezed-out distribution function depends only on

$$\frac{p_c}{T_d} = \frac{p_{ph}(t)}{T_d(t)}, \quad \text{where} \quad T_d(t) \equiv \frac{T_d}{a(t)}.$$

That is, the distribution function for DM particles that decouple in thermal equilibrium takes the form

$$F_d^{\rm equil}\left[\frac{p_{ph}(t)}{T_d(t)}\right] = F_d^{\rm equil}\left[\frac{p_c}{T_d}\right],$$

where $F_d^{\rm equil}$ is a Bose–Einstein or Fermi–Dirac distribution function:

$$F_d^{\rm equil}[p_c] = \frac{1}{\exp\left[\sqrt{m^2 + p_c^2}\Big/T_d\right] \pm 1}. \tag{45}$$

Note that for equation (45) in this regime

$$\frac{\sqrt{m^2 + p_c^2}}{T_d} \stackrel{T_d \gg m}{=} y + \mathcal{O}\left(\frac{m^2}{T_d^2}\right),$$

where y is defined by equation (6) and we can use as distribution functions

$$F_d^{\rm equil}(y) = \frac{1}{e^y \pm 1}. \tag{46}$$

Using equations (8) and (45), we find for fermions and for bosons decoupling at LTE that

$$m = \frac{g_d}{g}\begin{cases} 3.874 \text{ eV fermions} \\ 2.906 \text{ eV bosons}\,. \end{cases} \tag{47}$$

We see that for DM that decoupled at the Fermi scale, namely $T_d \sim 100$ GeV and $g_d \sim 100, m$ results in the keV scale, as noted by Bond & Szalay (1983), Pagels & Primack (1982) and Bond, Szalay & Turner (1982). DM particles may decouple earlier with $T_d > 100$ GeV, but g_d is always in the hundreds, even in grand unified theories (GUTs) in which T_d can reach the GUT energy scale. Therefore, equation (47) strongly suggests that the mass of the DM particles that decoupled UR in LTE is in the keV scale.

It should be noted that the Lee–Weinberg (Lee & Weinberg 1977; Sato & Kobayashi 1977; Vysotsky, Dolgov & Zeldovich 1977) lower bound as well as the Cowsik–McClelland (Cowsick & McClelland 1972) upper bound follow from equation (8), as shown in Boyanovsky et al. (2008a).

Computing the integrals in equation (13) with the distribution functions of equation (45) gives, for DM decoupling UR in LTE, that

$$\mathcal{D} = g\begin{cases} \frac{1}{4\pi^2}\sqrt{\frac{\zeta^5(3)}{15\zeta^3(5)}} = 1.9625 \times 10^{-3} \text{ fermions} \\ \frac{1}{8\pi^2}\sqrt{\frac{\zeta^5(3)}{3\zeta^3(5)}} = 3.6569 \times 10^{-3} \text{ bosons}, \end{cases} \tag{48}$$

where $\zeta(3) = 1.2020569\ldots$ and $\zeta(5) = 1.0369278\ldots.$

Inserting the distribution function equation (46) into equations (19) and (21) for m and g_d, respectively, we obtain

$$m = \left(\frac{Z}{g}\right)^{1/4} \text{keV}\begin{cases} 0.568 \text{ fermions} \\ 0.484\text{bosons}\,, \end{cases}$$

$$g_d = g^{3/4}\, Z^{1/4}\begin{cases} 155 \text{ fermions} \\ 180\text{bosons}\,. \end{cases} \tag{49}$$

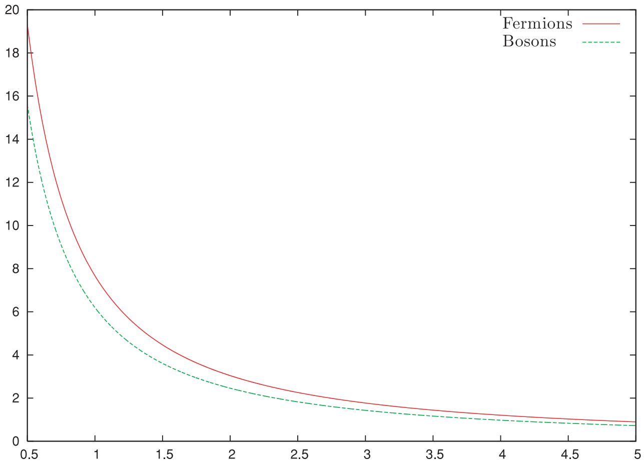

Figure 1. The free-streaming wavelength today, $\lambda_{\rm fs}(0)$, in kpc versus the DM particle mass, m, in keV times $g^{1/4}$ for ultrarelativistic decoupling at LTE according to equation (51).

Because $g = 1$–4, for DM particles decoupling at LTE, we see from equation (49) that $g_{\rm d} > 100$, and thus the DM particles should decouple for $T_{\rm d} > 100$ GeV. Note that $1 < Z^{1/4} < 10$ for $1 < Z < 10\,000$.

A further estimate for the DM mass m follows by inserting equation (48) for $\mathcal{D}$ in equation (24):

$$m \sim \left[\frac{\eta Z}{g}\;\frac{10\ {\rm km\,s^{-1}}}{\sigma_{\rm s}}\right]^{1/4}\sqrt{\frac{\rm kpc}{L}}\ {\rm keV}\begin{cases}0.405 \text{ fermions}\\ 0.347 \text{ bosons.}\end{cases} \quad (50)$$

Taking into account the observed values for $\sigma_{\rm s}$ and L from equation (23) and the fact that $\eta \lesssim 1.6, g \simeq 1$–$4, 1 < Z^4 < 10$, equation (50) gives again a mass m in the keV scale, as in equation (49). Both equations (49) and (50) yield a mass for the fermion 17 per cent greater than that for the boson.

We can express the free-streaming wavelength as a function of the DM particle mass from equations (28) and (49):

$$\lambda_{\rm fs}(z) = \left(\frac{\rm keV}{m}\right)^{4/3}\frac{\rm kpc}{g^{1/3}}\;\frac{1}{\sqrt{1+z}}\begin{cases}7.67 \text{ fermions}\\ 6.19 \text{ bosons.}\end{cases} \quad (51)$$

In Fig. 1 we show $\lambda_{\rm fs}(0)$ in kpc versus $m g^{1/4}$ in keV.

3.2 Decoupling out of LTE

In general, for DM decoupling out of equilibrium, the DM particle distribution function takes the form

$$F_{\rm d}(p_{\rm c}) = F_{\rm d}\left(\frac{p_{\rm c}}{T_{\rm d}};\frac{m}{T_{\rm d}};\ldots\right). \quad (52)$$

Typically, thermalization is reached by the mixing of the particle modes and scattering between particles that redistributes the particles in phase space: the larger-momentum modes are populated by a *cascade* whose front moves towards the ultraviolet, akin to a direct cascade in turbulence, leaving in its wake a state of nearly LTE but with a *lower* temperature than that of equilibrium (Boyanovsky, Destri & de Vega 2004; Destri & de Vega 2006). Hence, for DM particles not yet at thermodynamical equilibrium at decoupling, their momentum distribution is expected to peak at smaller momenta as the ultraviolet cascade is not yet completed (Boyanovsky et al. 2004; Destri & de Vega 2006). The distribution function freezed-out of equilibrium can be then written as

$$F_{\rm d}^{\rm out\ of\ LTE}(p_{\rm c}) = F_0 F_{\rm d}^{\rm equil}\left[\frac{a(t)\ p_{\rm ph}(t)}{\xi T_{\rm d}}\right]\ \theta\left(p_{\rm c}^0 - p_{\rm c}\right), \quad (53)$$

where $\xi = 1$ at thermal equilibrium and $\xi < 1$ before thermodynamical equilibrium is attained. $F_0 \sim 1$ is a normalization factor and $p_{\rm c}^0$ cuts the spectrum in the UV region not yet reached by the cascade.

Inserting the out-of-equilibrium distribution equation (53) into the expression for the DM particle mass equation (8) and using equation (46), we obtain the generalization of equation (47) for the out-of-LTE case:

$$m = \frac{g_{\rm d}}{g F_0 \xi^3}\ {\rm eV}\begin{cases}3.593\ \frac{F_+(\infty)}{F_+(s)} \text{ fermions}\\ 2.695\ \frac{F_-(\infty)}{F_-(s)} \text{ bosons,}\end{cases} \quad (54)$$

where $s = p_{\rm c}^0/[\xi T_{\rm d}]$. Here we used equation (46) and

$$F_\pm(s) \equiv \int_0^s \frac{y^2\,{\rm d}y}{{\rm e}^y \pm 1},\quad F_+(\infty) = \frac{3}{2}\zeta(3),\quad F_-(\infty) = 2\zeta(3). \quad (55)$$

Inserting the out-of-equilibrium distribution equation (53) into equations (19) and (21) for m and $g_{\rm d}$, respectively, and using equation (46), we obtain the estimates

$$m \sim \left(\frac{Z}{g}\right)^{1/4} W_\pm(s)\ {\rm keV}\begin{cases}0.568 \text{ fermions}\\ 0.484 \text{ bosons,}\end{cases} \quad (56)$$

$$r g_{\rm d} \sim g^{3/4}\ Z^{1/4}\xi^3 X_\pm(s)\begin{cases}155 \text{ fermions}\\ 180 \text{ bosons,}\end{cases} \quad (57)$$

where

$$W_\pm(s) \equiv \left[\frac{G_\pm^3(s)F_\pm^5(\infty)}{G_\pm^3(\infty)F_\pm^5(s)}\right]^{1/8},\quad X_\pm(s) \equiv \left[\frac{G_\pm(s)F_\pm(s)}{G_\pm(\infty)F_\pm(\infty)}\right]^{3/8}. \quad (58)$$

Here, $F_\pm(s)$ is defined by equation (55) and

$$G_\pm(s) \equiv \int_0^s \frac{y^4{\rm d}y}{{\rm e}^y \pm 1},\quad G_+(\infty) = \frac{45}{2}\zeta(5),\quad G_-(\infty) = 24\zeta(5). \quad (59)$$

For small arguments s we have:

$$W_+(0) = \frac{\sqrt{3}}{5^{3/4}}\left[\frac{\zeta^5(3)}{\zeta^3(5)}\right]^{1/8} = 0.5732982\ldots,$$

$$W_-(s) \overset{s\to 0}{=} \frac{s^{1/4}}{2^{5/8}}\left[\frac{\zeta^5(3)}{27\zeta^3(5)}\right]^{1/8} = 0.4753169\ldots\ s^{1/4},$$

$$X_+(s) \overset{s\to 0}{=} \frac{s^3}{[2025\zeta(3)\zeta(5)]^{3/8}} = 0.0529923\ldots\ s^3,$$

$$X_-(s) \overset{s\to 0}{=} \frac{s^{9/4}}{[4320\zeta(3)\zeta(5)]^{3/8}} = 0.0398856\ldots\ s^{9/4}.$$

As seen in Fig. 2,

$$W_\pm(s) \le 1 \quad\text{and}\quad X_\pm(s) \le 1 \quad\text{for}\quad s \ge 0.$$

We see from equation (56) that for relics decoupling out of LTE, m is in the keV range. From equations (56)–(57) we see that both m and $g_{\rm d}$ for relics decoupling out of LTE are smaller than they would be if they decoupled in LTE. In addition, as $X_\pm(s)$ vanishes for $s \to 0$, $g_{\rm d}$ may be much smaller than for decoupling in LTE.

We now generalize equation (48) for the phase-space density $\mathcal{D}$ to the out-of-LTE case (equation 53). Using equations (13), (46) and (53) we have

$$\mathcal{D} = g\ \frac{F_0}{W_\pm^4(s)}\begin{cases}1.9625 \times 10^{-3} \text{ fermions}\\ 3.6569 \times 10^{-3} \text{ bosons.}\end{cases} \quad (60)$$

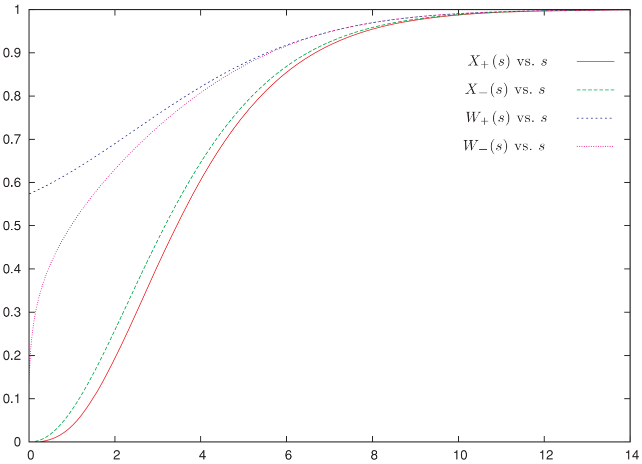

Figure 2. $X_+(s)$, $X_-(s)$, $W_+(s)$ and $W_-(s)$ as functions of s according to equation (58).

Inserting equation (60) into equation (24) leads to the out-of-LTE generalization of the estimate for the DM particle mass m (equation 50):

$$m \sim \left[\frac{\eta Z}{g}\,\frac{10\ \mathrm{km\,s^{-1}}}{\sigma_s}\right]^{1/4} W_\pm(s)\,\sqrt{\frac{\mathrm{kpc}}{L}}\ \mathrm{keV}\begin{cases}0.405 \text{ fermions}\\ 0.347 \text{ bosons.}\end{cases} \tag{61}$$

Taking into account the observed values for σ_s and L from equation (23) and the fact that $\eta \lesssim 1.6, g \simeq 1$–$4, 1 < Z^{1/4} < 10$, equation (61) gives again a mass m in the keV scale, as in equation (56). Equations (56) and (61) both yield a mass for the fermion 17 per cent larger than that for the boson.

3.2.1 An instructive example: sterile neutrinos decoupling out of LTE

We consider in this subsection a sterile neutrino ν as a DM particle decoupling out of LTE in a specific model in which ν is a singlet Majorana fermion ($g = 2$) with a Majorana mass m_ν coupled with a small Yukawa-type coupling ($Y \sim 10^{-8}$) to a real scalar field χ (Chikashige, Mohapatra & Peccei 1981; Gelmini & Roncadelli 1981; Schechter & Valle 1982; Shaposhnikov & Tkachev 2006; McDonald & Sahu 2009). χ is more strongly coupled to the particles in the Standard Model (plus to three right-handed neutrinos) than to υ. As a result, all particles (except ν) remain in LTE well after ν decouples from them.

The distribution function after the decoupling of the sterile neutrino ν is known for small coupling Y to be

$$F_d^\nu(y) = \tau\,\frac{g_{5/2}(y)}{\sqrt{y}}\,, \qquad \text{where} \qquad g_{5/2}(y) \equiv \sum_{n=1}^{\infty}\frac{e^{-n\,y}}{n^{5/2}} \tag{62}$$

(Boyanovsky 2008), and the coupling τ is in the range $0.035 \lesssim \tau \lesssim 0.35$ (Boyanovsky 2008).

It is interesting to compare the small ($y \to 0$) and large ($y \to \infty$) momentum behaviour of this out-of-equilibrium distribution $F_d^\nu(y)$ with the Fermi–Dirac equilibrium distribution (equation 46). We find that

$$\frac{F_d^\nu(y)}{F_d^{\mathrm{equil}}(y)} \overset{y\to 0}{=} \frac{2\tau\zeta(5/2)}{\sqrt{y}} \to \infty, \quad \zeta(5/2) = 1.341\ldots,$$
$$\frac{F_d^\nu(y)}{F_d^{\mathrm{equil}}(y)} \overset{y\to\infty}{=} \frac{\tau}{\sqrt{y}} \to 0\,.$$

Therefore, $F_d^\nu(y)$ exhibits an enhancement compared with the Fermi–Dirac equilibrium distribution for small ($y \to 0$) and a suppression for large ($y \to \infty$) momenta. Qualitatively, the out-of-equilibrium distribution equation (53) exhibits the same enhancement and suppression effects when compared with the equilibrium distribution F_d^{equil} as a consequence of the incomplete UV cascade.

We now evaluate the relevant physical quantities, inserting $F_d^\nu(y)$ into the appropriate equations of Section 3.1. We find m_ν from equations (8) and (62):

$$m_\nu = 2.34\,\frac{g_d}{\tau}\ \mathrm{eV}, \tag{63}$$

which must be compared with the LTE result for fermions (equation 47) with $g = 2$.

The phase-space density $\mathcal{D}$ from equations (13) and (62) takes the value

$$\mathcal{D} = \frac{6\tau\zeta^{5/2}(5)}{[35\pi\zeta(7)]^{3/2}} = 5.627\times 10^{-3}\,\tau\,,$$
$$\text{where}\quad \zeta(7) = 1.0083493\ldots\,. \tag{64}$$

This result is to be compared with the LTE result for fermions, namely equation (48) with $g = 2$.

Inserting the sterile neutrino distribution function (equation 62) into equations (19) and (21), which take into account the decrease of the phase-space density owing to the self-gravity dynamics, we obtain the following mass estimates for the ν DM particles that decoupled out of LTE:

$$m_\nu \sim \left(\frac{Z}{\tau}\right)^{1/4} 0.434\ \mathrm{keV}\,, \qquad g_d \sim \tau^{3/4} Z^{1/4}\, 185. \tag{65}$$

Again, these formulae must be compared with the LTE result for fermions (equations 49). $g_d \sim 100$ corresponds to $T_d \sim 100$ GeV (see Kolb & Turner 1990), which is the expected value for T_d in Boyanovsky (2008).

More precisely, for the typical range $0.035 \lesssim \tau \lesssim 0.35$, from equation (65) we find

$$0.56\ \mathrm{keV} \lesssim m_\nu Z^{-1/4} \lesssim 1.0\ \mathrm{keV}, \qquad 15 \lesssim g_d Z^{-1/4} \lesssim 84,$$

whereas for $g = 2$ fermions decoupling in LTE, the mass turns out to be smaller, $mZ^{-1/4} = 0.48$ keV, and g_d larger, $g_d Z^{-1/4} = 184$ (from equation 49).

A further estimate for m_ν, independent of equation (65), follows by inserting equation (64) for $\mathcal{D}$ into equation (24), valid for a self-gravitating gas of DM:

$$m_\nu \sim 0.3105\left[\frac{\eta Z}{\tau}\,\frac{10\ \mathrm{km\,s^{-1}}}{\sigma_s}\right]^{1/4}\sqrt{\frac{\mathrm{kpc}}{L}}\ \mathrm{keV}, \tag{66}$$

which gives for the typical τ range,

$$0.40\ \mathrm{keV} \lesssim m_\nu \left[\eta Z\,\frac{10\ \mathrm{km\,s^{-1}}}{\sigma_s}\right]^{-1/4}\sqrt{\frac{L}{\mathrm{kpc}}} \lesssim 0.72\ \mathrm{keV}$$

(Recall that $0.1 < Z^{-1/4} < 1$.)

In summary, the results for the sterile neutrino decoupling out of LTE in the model of Chikashige et al. (1981), Gelmini & Roncadelli (1981), Schechter & Valle (1982), Shaposhnikov & Tkachev (2006), Boyanovsky (2008) are qualitatively similar to those for fermions decoupling in LTE.

4 DM PARTICLES DECOUPLING NON-RELATIVISTICALLY

Particles decoupling NR at a temperature $T_d \ll m$ are described by a freezed-out Maxwell–Boltzmann distribution function depending on

$$\frac{p_c^2}{T_d} = \frac{a^2(t)\; p_{ph}^2(t)}{T_d} = T_d\; y^2.$$

That is,

$$F_d^{equil}(p_c) = \frac{2^{5/2}\,\pi^{7/2}}{45}\, g_d Y_\infty \left(\frac{T_d}{m}\right)^{3/2} e^{-\frac{p_c^2}{2mT_d}}$$
$$= \frac{2^{5/2}\pi^{7/2}}{45}\, g_d Y_\infty \left(\frac{T_d}{m}\right)^{3/2} e^{-\frac{a^2(t)p_{ph}^2(t)}{2mT_d}}$$
$$= \frac{2^{5/2}\pi^{7/2}}{45}\, \frac{g_d Y_\infty}{x^{3/2}}\, e^{-\frac{y^2}{2x}}\ , \tag{67}$$

where g_d is the effective number of ultrarelativistic degrees of freedom at decoupling and $Y(t) = n(t)/s(t)$, where $n(t)$ is the number of DM particles per unit volume and $s(t)$ their entropy per unit volume, $x \equiv m/T_d$ and Y_∞ follows from the late time limit of the Boltzmann equation (Kolb & Turner 1990; Börner 2003).

For particles that decouple NR we obtain, by inserting equation (67) into the general formula for the DM particle mass (equation 8), that

$$m = \frac{45}{4\pi^2}\,\frac{\Omega_{DM}\;\rho_c}{g T_\gamma^3 Y_\infty} = \frac{0.748}{g Y_\infty}\ \mathrm{eV}. \tag{68}$$

Solving the Boltzmann equation gives the following expression for Y_∞:

$$Y_\infty = \frac{45}{4\sqrt{2}\pi^{7/2}}\,\frac{g}{g_d}\, x e^{-x} \tag{69}$$

(Kolb & Turner 1990; Börner 2003). Note that $x \gtrsim 1$ because the DM particles decouple NR. Y_∞ can also be expressed in terms of σ_0 (the thermally averaged total annihilation cross-section times the velocity that appears in the Boltzmann equation) as (Kolb & Turner 1990; Börner 2003)

$$Y_\infty = \frac{1}{\pi}\sqrt{\frac{45}{8}}\,\frac{x}{\sqrt{g_d}\, m \sigma_0 M_{Pl}}. \tag{70}$$

(We assume for simplicity S-wave annihilation.) It follows from this relationship and equation (68) that

$$\sigma_0 = \frac{0.414\ 10^{-9}}{\mathrm{GeV}^2}\,\frac{g x}{\sqrt{g_d}}. \tag{71}$$

The transcendental equations (68) and (69) fix the values of m and x. They can be combined as

$$\frac{e^x}{x} = 193.5\,\frac{g^2}{g_d}\,\frac{m}{\mathrm{keV}}. \tag{72}$$

This equation has solutions for $x > 1$ provided that

$$\frac{m}{\mathrm{keV}} > \frac{e}{193.5}\,\frac{g_d}{g^2} = 0.014\,\frac{g_d}{g^2}.$$

For $x = m/T_d \gtrsim 1$ we have the analytic solution of equation (72):

$$\frac{m}{T_d} = x \simeq \log\left(193.5\,\frac{g^2}{g_d}\,\frac{m}{\mathrm{keV}}\right) = 5.265 + \log\left(\frac{g^2}{g_d}\,\frac{m}{\mathrm{keV}}\right).$$

We obtain the mass of the DM particle by inserting the non-relativistic distribution function (equation 67) into the general formula (equation 19) for m^4, with the result

$$m^{5/2} T_d^{3/2} = \frac{45}{2\pi^2}\,\frac{1}{g g_d Y_\infty}\, Z\,\frac{\rho_s}{\sigma_s^3}. \tag{73}$$

Combining equation (68) for Y_∞ with equation (73), we obtain for the product $m T_d$

$$\sqrt{m T_d} = 1.47\left(\frac{Z}{g_d}\right)^{1/3}\ \mathrm{keV}\quad \text{NR Maxwell–Boltzmann.} \tag{74}$$

Typical wimps are assumed to have $m = 100$ GeV and $T_d = 5$ GeV (Amsler et al. 2008). Such a value for T_d implies that $g_d \simeq 80$ (Kolb & Turner 1990). Equation (74) thus requires for such heavy wimps that $Z \sim 10^{23}$, well above the upper bounds derived in Section 2 (see Table 1). Therefore, wimps in the 100-GeV scale are strongly disfavoured.

We find from equations (13) and (67) the phase-space density for DM decoupling NR in LTE:

$$\mathcal{D} = g\,\frac{2\pi^2}{135\sqrt{3}}\, g_d Y_\infty \left(\frac{T_d}{m}\right)^{3/2} = 8.4418\times 10^{-2}\ g\ g_d\ Y_\infty\ x^{-3/2}\ . \tag{75}$$

We obtain, using the value for Y_∞ from equation (68), that

$$\mathcal{D} = 0.6315\times 10^{-4} g_d\,\frac{\mathrm{keV}}{m^{5/2}}\, T_d^{3/2}. \tag{76}$$

Inserting this expression for $\mathcal{D}$ into the general estimate for the DM mass (equation 24) yields

$$\sqrt{m T_d} \sim 0.942\left(\frac{\eta Z}{g_d}\right)^{1/3}\left(\frac{\mathrm{kpc}}{L}\right)^{2/3}\left(\frac{10\ \mathrm{km/s}}{\sigma_s}\right)^{1/3}\ \mathrm{keV}. \tag{77}$$

As in equation (74), but independently from it, we reach a result for $\sqrt{m T_d}$ in the keV scale, assuming that the DM is a self-gravitating gas in thermal equilibrium.

4.1 Allowed ranges for m, T_d and the annihilation cross-section σ_0

We derive here individual bounds on m, T_d and σ_0 for DM particles decoupling NR.

Using that $T_d < m$ for DM particles that decouple NR we obtain from equation (74) a lower bound for m and an upper bound on T_d. Furthermore, taking into account that $T_d > b$ eV, where $b > 1$ or $b \gg 1$ for DM particles that decoupled in the RD era, we obtain an upper bound for m. In summary,

$$\left(\frac{Z}{g_d}\right)^{1/3} 1.47\ \mathrm{keV} < m < \frac{2.16}{b}\ \mathrm{MeV}\ \left(\frac{Z}{g_d}\right)^{2/3}\ ,$$
$$b\ \mathrm{eV} < T_d < \left(\frac{Z}{g_d}\right)^{1/3} 1.47\ \mathrm{keV}\ . \tag{78}$$

Recalling that (Kolb & Turner 1990)

$$g_d \simeq 3 \quad \text{for} \quad 1\ \mathrm{eV} < T_d < 100\ \mathrm{keV}, \tag{79}$$

and that $1 < Z < 10^4$, we see from equations (78) that

$$1.02\ \mathrm{keV} < m < \frac{482}{b}\ \mathrm{MeV}, \quad T_d < 10.2\ \mathrm{keV}.$$

Note that b may be much larger than one with $b < 1470\ (Z/g_d)^{1/3} < 21960$ to ensure $T_d < m$ for consistency in equations (78).

In addition, lower and upper bounds for the cross-section σ_0 can be derived. From equations (71), (79) and $x > 1$ a lower bound follows:

$$\sigma_0 > 0.239\times 10^{-9}\ \mathrm{GeV}^{-2}\ g. \tag{80}$$

Upper bounds for the total DM self-interaction cross-sections σ_T have been given by comparing X-ray, optical and lensing observations of the merging of galaxy clusters with N-body simulations in Markevitch et al. (2004), Randall et al. (2008), Bradač et al. (2008) (see also Miralda-Escudé 2002; Hennawi & Ostriker 2002; Arabadjis, Bautz & Garmire 2002):

$$\frac{\sigma_T}{m} < 0.7\ \frac{\mathrm{cm}^2}{\mathrm{gr}} = 3200\ \mathrm{GeV}^{-3}.$$

Because the annihilation cross-section must be smaller than the total cross-section we can write the bound

$$\sigma_0 < 3200m \text{ GeV}^{-3}. \tag{81}$$

Using the upper bound (equation 78) for m yields the upper bound for σ_0:

$$\sigma_0 < \frac{3.32Z^{2/3}}{b} \text{ GeV}^{-2}. \tag{82}$$

This result leaves at least five orders of magnitude between the lower bound (equation 80) and the upper bound for σ_0. The DM non-gravitational self-interaction is therefore negligible in this context.

Exotic models in which very heavy ($\sim$10 TeV) DM particles are produced very late and decouple NR have been proposed, with the introduction two new fine-tuned parameters: (i) the lifetime of unstable particles (sneutrinos) that decay into DM (gravitinos); (ii) the mass difference between the two particles, which must be small enough to lead to non-relativistic DM (Cembranos et al. 2005; Strigari, Kaplinghat & Bullock 2007). It is argued in Cembranos et al. (2005) and Strigari et al. (2007) that such models may describe the observed phase-space density. It is pointed out in Borzumati, Bringmann & Ullio (2008) that it is inherently difficult to fulfil all observational constraints in such models.

5 CONCLUSIONS

Our results are *independent* of the particle model that describes the DM. We consider DM particles that decouple both NR and UR and that decouple both in and out of LTE. Our analysis and results refer to the mass of the DM particle and the number of ultrarelativistic effective degrees of freedom when the DM particles decoupled. We do not make assumptions about the nature of the DM particle and we assume only that its non-gravitational interactions can be neglected in the present context (which is consistent with structure formation and observations).

For DM particles to explain the formation of galactic centre black holes, DM particles must be fermions with keV-scale mass (Munyaneza & Biermann 2006).

The mass for DM particles in the keV range is much larger than the temperature during the MD era, and hence DM is cold (CDM).

A possible CDM candidate in the keV scale is the sterile neutrino (Dodelson & Widrow 1994; Shi & Fuller 1999; Abazajian, Fuller & Patel 2001; Abazajian 2006; Munyaneza & Biermann 2006; Kusenko 2007), produced through its mixing and oscillation with an active neutrino species. Other putative CDM candidates in the keV scale are the gravitino (Gorbunov, Khmelnitsky & Rubakov 2008; Steffen 2009, and references therein), the light neutralino (Profumo 2008, and references therein) and the majoron (Lattanzi & Valle 2007).

In fact, many more extensions of the Standard Model of Particle Physics can be envisaged to include a DM particle with mass in the keV scale and weakly enough coupled to the Standard Model particles.

Lyman α forest observations provide indirect lower bounds on the masses of sterile neutrinos (Viel et al. 2005, 2007), whereas constraints from the diffuse X-ray background yield upper bounds on the mass of a putative sterile neutrino DM particle (Dolgov & Hansen 2002; Watson et al. 2006; Boyarsky, Nevalainen & Ruchayskiy 2007; Riemer-Sorensen, Hansen & Pedersen 2006; Riemer-Sorensen et al. 2007; Loewenstein, Kusenko & Biermann 2009). All these recent constraints are consistent with DM particle masses at the keV scale.

The DAMA/LIBRA collaboration has confirmed the presence of a signal in the keV range (Bernabei et al. 2008a). Whether this signal results from DM particles in the keV mass scale is still unclear (Bernabei et al. 2006, 2008b; Pospelov, Ritz & Voloshin 2008). On the other hand, the DAMA/LIBRA signals cannot be explained by a hypothetical wimp particle with mass $\gtrsim O(1)$ GeV, as this would be in conflict with previous wimp direct detection experiments (Aalseth et al. 2008; Fairbairn & Schwetz 2009; Hooper et al. 2009; Savage et al. 2009; Ahmed et al. 2009).

As discussed in Section 4, typical wimps with $m = 100$ GeV and $T_d = 5$ GeV (Amsler et al. 2008) would require a huge $Z \sim 10^{23}$, well above the upper bounds displayed in Table 1. Hence, wimps cannot reproduce the observed galaxy properties. In addition, recall that $Z \sim 10^{23}$ produces, from equation (28), an extremely short λ_{fs} today:

$$\lambda_{fs}(0) \sim 3.51 \times 10^{-4} \text{ pc} = 72.4 \text{ au}.$$

If the flyby anomaly is to be explained by DM, a keV-scale DM mass is preferred (Adler 2009).

Further evidence for the DM particle mass in the keV scale follows by comparing the observed value of the constant surface density of galaxies with the theoretical calculation from the linearized Boltzmann–Vlasov equation (de Vega & Sánchez 2009). Independent further evidence for the DM particle mass in the keV scale is given by Tikhonov et al. (2009) (see also Gilmore et al. 2007).

In summary, our analysis shows that DM particles decoupling UR in LTE have a mass m in the keV scale with $g_d \gtrsim 150$, as shown in Section 3.1. That is, decoupling happens at least at the 100-GeV scale. The values of m and g_d may be smaller for DM decoupling UR out of LTE than for decoupling UR in LTE (see Section 3.2). For DM particles decoupling NR in LTE ($T_d < m$) we find that $\sqrt{mT_d}$ is in the keV range. This is consistent with the DM particle mass in the keV range.

Note that the present uncertainty by one order of magnitude of the observed values of the phase-space density ρ_s/σ_s^3 affects the DM particle mass only through a power 1/4 of this uncertainty according to equations (19)–(20). Namely, by a factor $10^{1/4} \simeq 1.8$.

We find that the free-streaming wavelength (Jeans length) is *independent* of the nature of the DM particle except for the Z factor characterizing the decrease of the phase-space density through self-gravity (Section 1.1). The values found for the Jeans length and the Jeans mass for m in the keV scale are consistent with the observed small structure and with the masses of the galaxies, respectively.

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